

Hurwitz Spaces and Moduli Spaces as Ball Quotients via Pull-back

Brent R. Doran

Abstract

We define hypergeometric functions using intersection homology valued in a local system. Topology is emphasized; analysis enters only once, via the Hodge decomposition. By a pull-back procedure we construct special subsets \mathcal{S}_π , derived from Hurwitz spaces, of Deligne-Mostow moduli spaces $DM(n, \mu)$. Certain $DM(n, \mu)$ are known to be ball quotients, uniformized by hypergeometric functions valued in a complex ball (i.e., complex hyperbolic space). We give sufficient conditions for \mathcal{S}_π to be a subball quotient. Analyzing the simplest examples in detail, we describe ball quotient structures attached to some moduli spaces of inhomogeneous binary forms. This recovers in particular the structure on the moduli space of rational elliptic surfaces given by Heckman and Looijenga. We make use of a natural partial ordering on the Deligne-Mostow examples (which gives an easy way to see that the original list of Mostow, eventually corrected by Thurston, is in error), and so highlight two key examples, which we call the Gaussian and Eisenstein ancestral examples.

1 Introduction

A number of classical moduli spaces \mathcal{M} admit the structure of a locally symmetric space $\Gamma \backslash G/K$, where K is a maximal compact subgroup and Γ is a discrete subgroup of G . The identification is given by a Γ -invariant map $\Phi : G/K \rightarrow \mathcal{M}$ that descends to an isomorphism from $\Gamma \backslash G/K$ to \mathcal{M} .

$$\begin{array}{ccc} & G/K & \\ \Phi \swarrow & \downarrow / \sim & \\ \mathcal{M} & \xleftarrow{\cong} & \Gamma \backslash G/K \end{array}$$

In the language of the 19th century, $(G/K, \Phi)$ is a *uniformization* of \mathcal{M} . If Γ does not act freely, it is an *orbifold uniformization*. For example, when $G = PU(1, n)$ then G/K is the complex n -ball \mathbb{B}^n , or complex hyperbolic n -space, and so the uniformization endows \mathcal{M} with a complex hyperbolic metric (possibly with orbifold singularities). We call $\mathcal{M} \cong \Gamma \backslash \mathbb{B}^n$ a “ball quotient”.

In their seminal work on hypergeometric functions, Deligne and Mostow [4, 23, 5] proved that certain geometric invariant theory (GIT) moduli spaces $DM(n, \mu)$ of n points on \mathbb{P}^1 are ball quotients. Here the uniformizing group Γ is a monodromy representation of the spherical braid group on n strings, the ball is \mathbb{B}^{n-3} , and the uniformizing map Φ is the single-valued inverse to a map HG_μ built from multi-valued generalized hypergeometric functions on $DM(n, \mu)$.

$$\begin{array}{ccc} & \mathbb{B}^{n-3} & \\ \Phi_\mu \swarrow & \downarrow / \sim & \\ DM(n, \mu) & \xrightarrow{\cong} & \Gamma \backslash \mathbb{B}^{n-3} \end{array}$$

In this paper, we make use of an overlooked, essentially topological, property of hypergeometric functions defined via local systems. This yields a method for producing subball quotients of $DM(n, \mu)$ that have natural geometric interpretations in terms of moduli.

Intersection homology has several convenient attributes [3, 13, 14, 20]. One is that it comes with an intersection pairing whose signature can be computed using explicit cycles.

Here, this pairing defines a Hermitian form, Ψ , which only depends on a list μ of n fractions (associated with the n points on \mathbb{P}^1). Consequently, varying the coordinates of the points, i.e., moving through the moduli space, preserves the form. In particular, the monodromy group Γ preserves Ψ : so the data μ determine a *lattice* Λ over a ring of integers R , together with Γ acting as automorphisms of Λ . Thus μ in fact defines a Hermitian locally symmetric space. This procedure parallels the main approach, using ordinary and compactly supported cohomologies, of Deligne and Mostow.

Also, intersection homology is a bivariate functor that it is insensitive to points of trivial local monodromy. Consequently, given $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, and a rank 1 local system $l_T \rightarrow \mathbb{P}^1 \setminus T$, the intersection homology of the pull-back rank 1 local system $\pi^*(l_T) \rightarrow \mathbb{P}^1 \setminus \pi^{-1}(T)$ only “sees” the interesting points $S \subset \pi^{-1}(T)$ that contribute to monodromy. This is the feature that allows us to relate certain Hurwitz spaces to subball quotients.

Finally, intersection homology here admits a Hodge decomposition into orthogonal subspaces corresponding to holomorphic and anti-holomorphic forms. This decomposition encodes the only actual analysis to enter into what is otherwise a topological construction. We define the multi-valued map HG_μ to be the coordinate expression of the holomorphic section of a flat Grassmannian bundle (over the moduli space of n distinct points on \mathbb{P}^1) given by the linear subspace of holomorphic 1-forms. For the Deligne-Mostow examples, the flat Grassmannian bundle is actually a flat projective space bundle and Ψ is of Lorentzian signature, so that HG_μ is valued in a complex ball in projective space, as desired.

Section 2 develops the theory of hypergeometric functions of Deligne-Mostow type from the perspective of intersection homology valued in a local system. We take a hands-on approach to understanding the cycles, the form Ψ , and the monodromy group Γ . No claim is made as to original results, although some proofs may be new. The reader is referred to the original paper of Deligne and Mostow [4] for some technical details that carry over to intersection homology *mutatis mutandis*, and for a complete discussion of the theory of holomorphic and anti-holomorphic forms valued in a rank 1 local system on an n -punctured \mathbb{P}^1 . The section concludes by recalling the fundamental uniformization results of Deligne and Mostow [4, 23], via conditions INT and ΣINT .

Section 3 serves two purposes, both related to the Ψ -lattices that underlie the Deligne-Mostow uniformizations. First it establishes that two important examples, which we call the Eisenstein and Gaussian ancestral examples, have a rich supply of subball quotients. This is key for the results of Section 4. Second, the section makes concrete a fact which is implicit in Deligne and Mostow’s analysis, but is often overlooked. Namely, GIT-stable collisions of points on \mathbb{P}^1 in a Deligne-Mostow uniformization correspond to orbifold subball quotients (basically, the mirrors of complex reflections associated to the monodromy group Γ). This gives an easy way to organize much of the Deligne-Mostow list of uniformizations; for instance, when $n \geq 7$ every example but one is a collision “descendant” of an ancestral example. In particular, it is immediate that Mostow’s original list [23] is missing a number of examples. (Thurston corrected that list by computer in [27], and to the best of our knowledge his list is complete, although a proof is not provided.) We show in [8] (and it is also seen by somewhat different means in [6]) that one of these “missing” examples is a cover of the ball quotient structure on the moduli space of cubic surfaces from [2].

Section 4 discusses the pull-back procedure. Given a map $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and a collection of points T , let ν denote the data of the non-trivial local monodromy of a rank 1 local system $l_T \rightarrow \mathbb{P}^1 \setminus T$. Thus ν may be written as a list of $|T|$ fractions. The pull-back local system $\pi^*l_T \rightarrow \mathbb{P}^1 \setminus \pi^{-1}(T)$ is specified by its local monodromy list of fractions μ . Given ν , the data μ is determined entirely by the ramification of π over T . Consequently, varying π while preserving the ramification over T (describing a Hurwitz space of \mathbb{P}^1 covers of \mathbb{P}^1), results in a constrained variation of the points of $\pi^{-1}(T)$. The variation of the subset $S \subset \pi^{-1}(T)$ of non-trivial local monodromy points thus determines a subvariety \mathcal{S}_π of a Deligne-Mostow moduli space $DM(n, \mu)$. If this moduli space is a ball quotient, then one may ask if \mathcal{S}_π is a subball quotient via restriction of hypergeometric functions. We give a sufficiency criterion. Detailed classification in very special cases with $|T| = 3$ produces a list of \mathcal{S}_π that at once are codimension 1 subball quotients and admit natural finite covers by moduli spaces of inhomogeneous binary forms. The codimension 1 subball quotient of the Eisenstein ancestral example may be interpreted as the moduli space of rational elliptic surfaces, via the Weierstrass fibration description of Miranda [22], thus recovering the ball quotient structure on that space described in [16].

Throughout we assume the number of points on \mathbb{P}^1 is $n \geq 3$.

2 Hypergeometric Functions after Deligne and Mostow

2.1 Background on local systems and subsystems

Let X be a connected manifold. The following are equivalent characterizations of a complex local system $L \rightarrow X$ up to isomorphism:

1. complex vector bundle with flat connection
2. locally constant sheaf of complex vector spaces
3. $\pi_1(X, 0)$ representation on a complex vector space, known as the *monodromy* representation, where 0 is some chosen base point in X

To pass from the first description to the second, identify the flat vector bundle with its sheaf of locally constant sections. Fix a base point $0 \in X$. Then the functor “fiber at 0” produces a complex vector space endowed with an action of $\pi_1(X, 0)$, hence deriving the third description from the second. Note that the choice of base point does not affect the isomorphism class, since a different base point yields the same monodromy representation up to conjugacy in the general linear group of the vector space.

In the case of rank 1 local systems, the monodromy representation is one-dimensional hence abelian. Since $H_1(X)$ is naturally isomorphic to $\pi_1(X)^{ab}$, the local system is determined by a homomorphism $H_1(X) \rightarrow \mathbb{C}^*$.

The simplest case is also the basic object of study for this paper, namely, rank 1 local systems on the projective line punctured at n points $s_j, j \in \{1, \dots, n\}$. Observe $H_1(\mathbb{P}^1 \setminus S)$ is an abelian group. Take positively oriented circles centered at the s_j to be representative cycles of the group generators. The only relation is that the product of the generators is the identity. It is easy to show that:

Proposition 1. *Given a set of points $S = \{s_1, \dots, s_n\}$ in \mathbb{P}^1 and a set of complex numbers $\mu = \{\mu_1, \dots, \mu_n\}$, there is, up to isomorphism, a unique rank 1 local system on $\mathbb{P}^1 \setminus S$ with $(S, e^{2\pi i \mu})$ as the local monodromy data. However, the local system is not determined up to unique isomorphism; the fibers may be uniformly rescaled by any element of \mathbb{C}^* .*

If the μ_j are real then $\alpha_j = e^{2\pi i \mu_j}$ is on the unit circle in \mathbb{C}^* . In particular the local monodromy may be of finite order. For the purposes of the Deligne-Mostow theory of hypergeometric functions, one takes $\mu_j \in \mathbb{Q}, \forall j$. Ultimately this condition will follow from the constraints INT or ΣINT on μ that guarantee uniformization, so it is a matter of convenience to demand it in advance.

Definition 1. *A Deligne-Mostow local system is a rank 1 local system $L \rightarrow \mathbb{P}^1 \setminus S$, where S is a finite collection of points, such that for all j, μ_j (encoding the local monodromy data $\alpha_j = e^{2\pi i \mu_j}$ at s_j) is a rational number. Clearly the μ_j may be adjusted to lie between 0 and 1 without changing L .*

By a *local subsystem* of a local system L we mean a locally constant subsheaf. Note that the monodromy data of a DM local system is defined over the ring of integers $R = \mathbb{Z}[\zeta_d]$, where ζ_d is a d^{th} root of unity, and d is the least common denominator of the μ_j . We call the corresponding local subsystem with fiber R the *Deligne-Mostow local subsystem*, denoted $L(R)$.

The *dual local system* L^\vee will be needed for the homology theory of the next section. It has a straightforward explicit description.

Proposition 2. *If L is the DM local system determined by the data (S, α) , equivalently by (S, μ) , then L^\vee is the DM local system determined by $(S, \bar{\alpha})$, equivalently by $(S, 1 - \mu)$. In other words, $L^\vee = \bar{L}$. Furthermore $L^\vee(R) = \bar{L}(R)$, where R is the ring of integers defining the Deligne-Mostow local subsystem of L .*

Proof. By Proposition 1, both L and L^\vee are characterized by local monodromy data. Thus, if the data $(S, \{\alpha_j\})$ determine L , then L^\vee is characterized by $(S, \{\alpha_j^{-1}\})$. Furthermore, if L is a Deligne-Mostow local system then $\alpha_j^{-1} = \bar{\alpha}_j$, because α_j lies on the unit circle in \mathbb{C} . So L^\vee is determined by $\bar{\alpha}_j$, and hence by $-\mu_j$, or equivalently, by $1 - \mu_j$ (normalizing to lie between 0 and 1). It is immediate from the explicit local monodromy data that all of the corresponding DM local subsystems are defined over the same ring of integers R . \square

2.2 Intersection homology valued in Deligne-Mostow local systems

2.2.1 Background on intersection homology

What follows is an informal discussion. The goal is to impart intuition and to highlight the results needed in the sequel. Details for the trivial local system case can be found in [3, 13, 14], and the arguments are easily adapted for general local systems (see also [20]).

Intersection homology can be defined for any Whitney stratified pseudo-manifold. Any quasi-projective variety X admits a Whitney stratification, where the unique open stratum, $X^{nonsing}$ is the “nonsingular part” of X . Intersection homology is a topological invariant, independent of the choice of stratification. The simplest definition is the original formulation, due to Goresky and MacPherson, in terms of geometric chains. Many models for the chains are acceptable, but for our purposes piecewise linear chains are perfectly satisfactory.

Intersection homology theory is similar to ordinary homology theory on X . The boundary operator is the same but the intersection chain complex, $IC(X)$ is a subcomplex of the ordinary chain complex. Those ordinary chains whose intersection with the singular locus X^{sing} are too “perverse”, i.e., too non-generic, are disallowed. A choice \bar{p} of *perversity* is then a choice of which chains are admissible. The default choice of perversity for algebraic varieties is “middle perversity.” Middle perversity intersection homology of X , denoted $IH_*(X)$, has many nice properties — the so-called *Kähler package*. The most important property here is Poincaré-Verdier duality:

Proposition 3. (*Poincaré-Verdier duality*) *Let d be the real dimension of X . There is a non-degenerate bilinear pairing*

$$IH_k(X) \otimes IH_{d-k}(X) \xrightarrow{\cap} \mathbb{C}$$

Let us now be more precise. Let X^{sing} be stratified by $\{S_\beta\}$, where β is the codimension of S_β in X .

Definition 2. • A (classical) perversity \bar{p} is a positive integer-valued non-decreasing function on the natural numbers $\{2, \dots, \dim_{\mathbb{R}}(X)\}$, satisfying $\bar{p}(2) = 0$ and $\bar{p}(\beta + 1) \leq \bar{p}(\beta) + 1$.

- An i -chain ξ in X is an intersection i -chain if it satisfies the admissibility conditions:
 1. $\dim_{\mathbb{R}}(\xi \cap S_\beta) \leq \dim_{\mathbb{R}}(S_\beta) + \dim_{\mathbb{R}}(\xi) - n + \bar{p}(\beta)$
 2. $\dim_{\mathbb{R}}(\partial\xi \cap S_\beta) \leq \dim_{\mathbb{R}}(S_\beta) + \dim_{\mathbb{R}}(\partial\xi) - n + \bar{p}(\beta)$

Remark 1. The perversity starts with codimension 2 because the singular locus of a pseudo-manifold is real codimension at least 2. The second admissibility condition ensures the intersection chains form a complex.

Just as with homology, intersection homology can be valued in sheaves other than the constant sheaf. In particular one considers intersection homology valued in a complex local system $L \rightarrow X \setminus S$. The standard notation (using middle perversity) is $IH_*(X, L)$. Here X^{sing} is S . The support of an L -valued intersection chain is, as before, a geometric chain in X that satisfies admissibility conditions based on choice of perversity. The only difference is that, in the nonsingular locus $X^{nonsing}$, the “value” attached to the chain is a section of L over the chain. In other words, an L -valued intersection chain is an ordinary chain in $X^{nonsing}$, valued in L , for which the closure of its support satisfies the admissibility criteria to be an intersection chain in X .

When $L = \mathbb{C}$, the trivial rank 1 local system on X , one of course recovers the usual intersection homology with complex coefficients.

2.2.2 Vector space structure with basis

Now let $L \rightarrow \mathbb{P}^1 \setminus S$ be a DM local system.

Lemma 1. *The geometric support of $IC_0(\mathbb{P}^1, L)$ and of $IC_1(\mathbb{P}^1, L)$ is $\mathbb{P}^1 \setminus S$. Let $K \subset S$ be the subset of points of nontrivial monodromy. Then the geometric support of $IC_2(\mathbb{P}^1, L)$ is $\mathbb{P}^1 \setminus K$.*

Proof. All choices of (classical) perversity are equivalent for a one-dimensional complex variety, because $\bar{p}(2) = 0$. So the middle perversity is the zero perversity $\bar{p} \equiv 0$. Consequently,

the first admissibility criterion disallows both points $s_j \in S$ and any 1-chains that intersect an s_j , but imposes no constraint on the 2-chains. The second admissibility criterion does not restrict the 0- and 1-chains further.

The application of the second admissibility criterion to 2-chains is more subtle, because the chains are valued in a local system. There are two types of intersection with S : the 2-chain either contains an s_j with non-trivial monodromy ($\alpha_j \neq 1$) or it only contains s_j with trivial monodromy. In the latter case, the boundary of the 2-chain is a 1-cycle that encloses but does not intersect s_j , and so is admissible. In the former case, observe that any such L -valued 2-chain has as boundary a 1-cycle that intersects s_j and so is *not* admissible. \square

Example: Let C denote a small circle oriented counter-clockwise and centered at s_k , and let θ denote the line segment from s_k to p on C . Denote the choice of section of L at p by \hat{p} , and its horizontal extension over C and θ by \hat{C} and $\hat{\theta}$, respectively. This determines a unique horizontal section over the disk D , denoted \hat{D} , with discontinuities (when $\alpha_k \neq 1$) along θ .

$$\begin{array}{c} C \\ \times \\ s_k \end{array} \xrightarrow{\theta} p \Rightarrow \times \xrightarrow{(\alpha_k - 1)\hat{\theta}} \hat{p} \quad \sim \quad \begin{array}{c} \hat{C} \\ \times \\ \hat{p} \end{array}$$

It is easy to see that $\partial \hat{D} = \hat{C} - (\alpha_k - 1)\hat{\theta}$. In particular, the support of the boundary intersects s_k , violating the second admissibility criterion.

Intersection homology is insensitive to points s_j of trivial local monodromy. More precisely, any intersection 1-cycle enclosing such a point s_j is homologous to an intersection 1-cycle that does not enclose it. This homology is realized by an intersection 2-chain that contains s_j and takes values in the trivial local system (extended over s_j). Formalizing this argument yields the following Lemma.

Lemma 2. *Let $L \rightarrow \mathbb{P}^1 \setminus S$ be a rank 1 complex local system. Let K be the subset of points $\{s_{m_1}, \dots, s_{m_k}\} \subset S$ with nontrivial local monodromy, that is, those points s_j with $\alpha_j \neq 1$. Let \hat{L} denote the local system on $\mathbb{P}^1 \setminus K$ defined by the local monodromies α_{m_j} . Then there is a natural isomorphism $IH_1(\mathbb{P}^1, L) \cong IH_1(\mathbb{P}^1, \hat{L})$.*

Thus the study of intersection homology valued in Deligne-Mostow local systems reduces to considering those local systems defined by $\alpha_j \neq 1, \forall j$. In that case, it is elementary to prove that the first homology groups in all the usual homology theories are isomorphic.

Lemma 3. *Let K and \hat{L} be as above. Then there are natural isomorphisms*

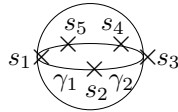
$$IH_1(\mathbb{P}^1, \hat{L}) \cong H_1(\mathbb{P}^1 \setminus K, \hat{L}) \cong H_1^{lf}(\mathbb{P}^1 \setminus K, \hat{L}),$$

where H_1^{lf} denotes locally finite homology.

Proposition 4. *Let $K = \{s_{m_1}, \dots, s_{m_k}\}$ be the subset of points s_j in S with $\alpha_j \neq 1$, and define $\hat{L} \rightarrow \mathbb{P}^1 \setminus K$ as before. Assume $K \neq \emptyset$. Then*

$$\dim_{\mathbb{C}}(IH_0(\mathbb{P}^1, \hat{L})) = 0, \dim_{\mathbb{C}}(IH_1(\mathbb{P}^1, \hat{L})) = k - 2, \dim_{\mathbb{C}}(IH_2(\mathbb{P}^1, \hat{L})) = 0.$$

Proof. Use the isomorphisms of Lemma 2 and Lemma 3 to identify $IH_1(\mathbb{P}^1, L)$ with $H_1^{lf}(\mathbb{P}^1 \setminus K, \hat{L})$. A good choice of k generators for $H_1^{lf}(\mathbb{P}^1 \setminus K, \hat{L})$ is the set of 1-cycles γ_j with endpoints $s_{m_j}, s_{m_{j+1}}$ (where the final one, γ_k , connects endpoints s_{m_k}, s_1 in that order). Without loss of generality, let the s_{m_j} be aligned along the equator, so that the γ_j themselves form the equator.



Relations among the homology generators are precisely those linear combinations of γ_j which are the boundary of some locally finite 2-chain \hat{D} . It is clear that the support of \hat{D}

must be either the upper or lower hemisphere. Pick a point p in the upper hemisphere (the choice of hemisphere is not important) and a section \hat{p} of \hat{L} over p . The section has a unique horizontal extension over the hemisphere containing p , so that $\partial\hat{D} = \sum_j \gamma_j$, so in homology

$$\sum_j \gamma_j = 0.$$

This is the first linear relation among the generators. Picking a different lift than \hat{p} simply rescales the section, and hence the boundary relation, by a complex number; so this choice doesn't alter the linear relation. The horizontal section extends to the lower hemisphere, but now a choice must be made: the natural extension is to a *multi-section* on $\mathbb{P}^1 \setminus K$. The choice therefore lies in selecting a γ_j over which to *continuously* extend the horizontal section into the lower hemisphere, to get a single-valued section; but it is not continuous along the remaining $\gamma_i, i \neq j$. By crossing at γ_j , the resulting \hat{D} (with D now the lower hemisphere) has a boundary that can be explicitly written in terms of local monodromies, yielding the second linear relation on the homology generators:

$$\alpha_2^{-1}\alpha_3^{-1}\cdots\alpha_j^{-1}\gamma_1 + \cdots \alpha_j^{-1}\gamma_{j-1} + \gamma_j + \alpha_{j+1}\gamma_{j+1} + \cdots \alpha_{j+1}\cdots\alpha_k\gamma_k = 0.$$

Observe that a different choice of γ_j simply rescales the linear combination by a complex number, so there is no change to the relation. Thus there are precisely two relations on the k generators, so $\dim_{\mathbb{C}}(IH_1(\mathbb{P}^1, \hat{L})) = k - 2$.

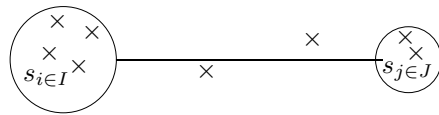
The other intersection homologies are easy to compute. $IH_0(\mathbb{P}^1, \hat{L}) = 0$ because for any point $p \in \mathbb{P}^1 \setminus K$ and section \hat{p} , the boundary of a 1-cycle whose support passes through p that loops around precisely one s_{m_j} is just $(\alpha_{m_j} - 1)\hat{p}$, and so \hat{p} is a boundary of an intersection 1-cycle. $IH_2(\mathbb{P}^1, \hat{L}) = 0$ because any 2-chain has non-trivial boundary so there are no 2-cycles. \square

Corollary 1. *Given any two points $s_i, s_j \in K$, the intersection 1-chain $I_{i,j} \equiv \frac{1}{\alpha_i - 1}\hat{C}_i + \hat{\gamma} + \frac{1}{1 - \alpha_j}\hat{C}_j$ is in fact a 1-cycle not homologous to zero. Moreover, given any partition of a subset of K into two disjoint collections $\{s_i\}_{i \in I}$ and $\{s_j\}_{j \in J}$ (I, J index sets such that $I \cap J = \emptyset$) where $\prod_{i \in I} \alpha_i \neq 1$ and $\prod_{j \in J} \alpha_j \neq 1$ (or equivalently, $\sum_{i \in I} \mu_i$ and $\sum_{j \in J} \mu_j \notin \mathbb{Z}$), the analogous 1-chain $I_{I,J}$ that encircles the two collections and connects them by a segment γ is in fact a 1-cycle.*

Proof. The locally finite 1-cycle with support a line segment $\gamma_{i,j}$ from s_i to s_j and section determined by extending \hat{p} at p is in the same locally finite homology class as the intersection 1-cycle $I_{i,j} \equiv \frac{1}{\alpha_i - 1}\hat{C}_i + \hat{\gamma} + \frac{1}{1 - \alpha_j}\hat{C}_j$. In particular, it is non-zero in locally finite homology. Because $IC_2(\mathbb{P}^1, \hat{L}) \subset C_2^{lf}(\mathbb{P}^1 \setminus K, \hat{L})$, it is clear $I_{i,j}$ cannot be the boundary of any intersection 2-chain, and so is non-zero in intersection homology.



The statement for collections of points $\{s_i\}_{i \in I}$ and $\{s_j\}_{j \in J}$ is immediate. Denote the closed curve that encircles the first collection (and no other s_l) by C_1 , mark a point $p \in C_1$ and a choice of section at p by \hat{p} that extends to a section \hat{C}_1 . Likewise about the second collection construct C_2 and q , and connect p and q with the line segment γ . Extend the horizontal section from \hat{p} to $\hat{\gamma}$ and \hat{C}_2 .



The boundary of the intersection 1-chain

$$I_{I,J} = \frac{1}{\prod_{i \in I} \alpha_i - 1} \hat{C}_1 + \hat{\gamma} + \frac{1}{1 - \prod_{j \in J} \alpha_j} \hat{C}_2$$

is zero, hence it is a 1-cycle. \square

We therefore get an intersection homology basis taken from the set $\{I_{i,i+1}\}_{i \in \{1, \dots, k-1\}} \cup \{I_{k,1}\}$.

Corollary 2. *Partition S into two subsets S_1 and S_2 , whose elements are indexed by i and j respectively. The cycles $\{I_{i,i+1} \equiv \frac{1}{\alpha_i - 1} \hat{C}_i + \hat{\gamma} + \frac{1}{1 - \alpha_{i+1}} \hat{C}_{i+1}\}_{i \in \{1, \dots, |S_1| - 1\}}$ taken together with the cycles $\{I_{j,j+1}\}_{j \in \{|S_1| + 1, \dots, |S_2| - 1\}}$ form a basis for $IH_1(\mathbb{P}^1, L)$ if and only if $\sum_{i \in S_1} \mu_i \notin \mathbb{Z}$ (or equivalently $\sum_{j \in S_2} \mu_j \notin \mathbb{Z}$).*

Remark 2. In fact, more is true. Given any partition of S into S_1 and S_2 (satisfying the above condition on the μ_i and μ_j), any tree connecting the points of S_1 taken together with any tree connecting the points of S_2 defines a basis in locally finite homology. This translates into a basis for intersection homology in the fashion indicated above. For a proof of the locally finite homology fact, see [4, Section 2.5]

2.2.3 Intersection pairing: skew-Hermitian form on $IH_1(\mathbb{P}^1, L)$

Proposition 5. *The data (S, μ) determine, up to a real scalar, the intersection pairing on intersection homology. The pairing puts a skew-Hermitian form on $IH_1(\mathbb{P}^1, L)$, so multiplication by the complex number $\iota = \sqrt{-1}$ yields an Hermitian form Ψ , unique up to a real scalar.*

Proof. Poincaré-Verdier duality gives a nondegenerate bilinear pairing between $IH_k(X, L)$ and $IH_{d-k}(X, L^\vee)$, where d is the real dimension of X and L^\vee is the dual local system. By Proposition 2, for a Deligne-Mostow local system $L^\vee = \overline{L}$. Because $IH_*(\overline{L}) = \overline{IH_*(L)}$, the duality pairing is

$$IH_k(X, L) \otimes IH_{d-k}(X, L^\vee) = IH_k(X, L) \otimes \overline{IH_{d-k}(X, L)} \xrightarrow{\cap} L \otimes L^\vee \cong \mathbb{C}$$

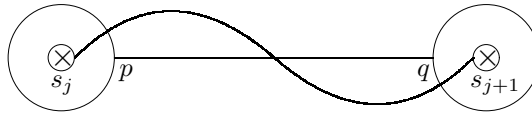
Thus there is a skew-Hermitian intersection form on $IH_1(X, L)$. From this, a Hermitian form Ψ is obtained by multiplication by ι . By Proposition 1, given $\{\mu_j\}$, L is determined up to a \mathbb{C}^* factor, so the intersection pairing and Hermitian form Ψ are determined up to a real scalar. \square

The intersection pairing on intersection 1-cycles, expressed in the basis of Corollary 2, is a skew-Hermitian matrix Int . Let $Int(i, j)$ denote the (i, j) entry of Int . If $|i - j| > 1$ then $Int(i, j) = 0$, because the support of $I_{i,i+1}$ doesn't intersect that of $I_{j,j+1}$. It remains to compute the self-intersection of $I_{i,i+1}$ and the intersection number for adjacent basis cycles (when $|i - j| = 1$).

Proposition 6. *The skew-Hermitian intersection form, with respect to the basis $I_{i,i+1}, i \in \{1, \dots, k-2\}$, is the matrix Int with entries:*

$$Int(i, j) = \begin{cases} \frac{1}{1 - \alpha_i} - 1 + \frac{1}{1 - \alpha_{i+1}} & , \quad j = i \\ -\frac{1}{1 - \alpha_i} & , \quad j = i + 1 \\ \frac{1}{1 - \alpha_i} & , \quad j = i - 1 \\ 0 & , \quad |j - i| > 1 \end{cases}$$

Proof. The computation is immediate from the following picture. The positive orientation is taken to be counterclockwise.



Note that the deformation chosen to compute the intersection number is particularly convenient given the choice of section (with discontinuities at p and q). A different choice would, of course, yield the same number, albeit presented as a sum of different terms. \square

The Hermitian form Ψ is simply ι times the intersection pairing, so in matrix form, $\Psi(j, k) = \iota \text{Int}(j, k)$.

Example: When $n = 4$,

$$\Psi = \iota \begin{bmatrix} \frac{1}{1-\alpha_1} - 1 + \frac{1}{\alpha_2} & -\frac{1}{1-\alpha_2} \\ \frac{1}{1-\alpha_2} & \frac{1}{1-\alpha_2} - 1 + \frac{1}{\alpha_3} \end{bmatrix}$$

In particular, if $\mu_i = \frac{1}{2}, \forall i$, then

$$\Psi = \begin{bmatrix} 0 & -\frac{\iota}{2} \\ \frac{\iota}{2} & 0 \end{bmatrix}$$

One application is that the signature of the form can be computed purely in terms of $\sum_i \mu_i$. There are a number of ways to show this. We give a constructive argument, which produces an explicit basis for a maximal positive definite subspace and its orthogonal negative definite subspace in $IH_1(\mathbb{P}^1, \hat{L})$.

Let $I_{I,J}$ be the intersection 1-cycle described above, enclosing $s_i, i \in I$ with C_1 and $s_j, j \in J$ with C_2 . Recall that the $\mu_i \in (0, 1)$. The monodromy along C_1 is given by $\prod_{i \in I} \alpha_i$, and so is determined by the fractional part of $\sum_{i \in I} \mu_i$. The analogous statement holds for the monodromy along C_2 .

Lemma 4. *Let $\text{Frac}(x)$ denote the fractional part of the non-negative real number x , i.e., $x - \lfloor x \rfloor$. If the sum $\text{Frac}(\sum_{i \in I} \mu_i) + \text{Frac}(\sum_{j \in J} \mu_j) < 1$ then the length of $I_{I,J}$ is negative, that is, $\Psi(I_{I,J}, I_{I,J}) < 0$. If $\text{Frac}(\sum_{i \in I} \mu_i) + \text{Frac}(\sum_{j \in J} \mu_j) = 1$ then $\Psi(I_{I,J}, I_{I,J}) = 0$. If $\text{Frac}(\sum_{i \in I} \mu_i) + \text{Frac}(\sum_{j \in J} \mu_j) > 1$ then $\Psi(I_{I,J}, I_{I,J}) > 0$.*

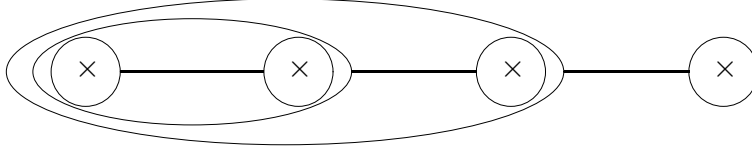
Proof. This is just clever work with the self-intersection number computed above:

$$-\frac{1}{\prod_{i \in I} \alpha_i - 1} - 1 + \frac{1}{1 - \prod_{j \in J} \alpha_j}$$

□

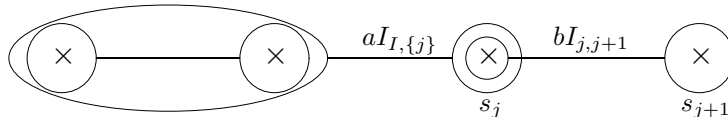
Proposition 7. *The signature of Ψ is $(\sum \mu_i - 1, \sum(1 - \mu_i) - 1)$ where the first term is the dimension of a maximal positive definite subspace and the second is the dimension of a maximal negative definite subspace.*

Proof. The strategy is to build a succession of $I_{I,J}$ which are mutually orthogonal. Since the sign of the Ψ -length of $I_{I,J}$ is known by Lemma 4, we get explicit maximal positive definite and negative definite subspaces. We inductively construct $I_{I,\{j\}}$ as shown in the following picture.



In particular, assuming none of these is zero length, for each positive integer less than $\sum_i \mu_i$ we produce a new positive length vector in the positive definite subspace, linearly independent from the preceding $I_{I,J}$. All of the remaining basis vectors generated by this procedure are in the orthogonal negative definite subspace. Because the total dimension is $n - 2$, we get the stated result.

Now assume that some $I_{I,\{j\}}$ so constructed has zero length. Select a and b so that $aI_{I,\{j\}} + bI_{\{j\},\{j+1\}}$ has positive length. Then $\bar{a}I_{I,\{j\}} + \bar{b}I_{\{j,j+1\}}$ is orthogonal and has negative length. Furthermore these are orthogonal to all previously constructed vectors in the sequence. In particular the number of positive and negative definite vectors produced remains unchanged. Continue the inductive procedure as before.



2.2.4 Lattice structure over ring of integers

We now recall the notion of a lattice from the theory of modules. The ring R will always denote a ring of algebraic integers. We will study the structure in greater depth in Section 3.

Definition 3. A module-theoretic lattice Λ is a finite rank module over a ring R , endowed with an Hermitian form Ψ taking values in R . A lattice is unimodular if the determinant of Ψ (the discriminant of Λ) is ± 1 . A sublattice M of Λ is a submodule together with the restriction of Ψ . A sublattice M is primitive if there is no other sublattice M' such that $M = rM'$ for r not a unit in R . The group $\text{Aut}(\Lambda)$ of unitary transformations (equivalently, lattice automorphisms) of Λ is the group of module automorphisms of Λ that preserve Ψ .

The intersection homology valued in a DM local system has the structure of a lattice.

Lemma 5. Let R be the ring of integers in $\mathbb{Q}[\zeta_d]$, where d is the lowest common denominator of the μ_i , and ζ_d is a primitive d^{th} root of unity. Then Ψ is defined over R .

Proof. It is immediate from our matrix descriptions of Ψ that it is defined over R . More formally, this follows by considering the local subsystem $L(R)$ with fiber the subring $R \subset \mathbb{C}$. The pairing of $L(R)$ with $L^{-1}(R) = \overline{L(R)}$ induces an R -valued skew-Hermitian pairing on $IH_1(\mathbb{P}^1, L(R))$. This pairing may be identified with the pairing from Section 2.2.3, since it was there only determined up to a real scalar by the data μ . □

2.3 Configuration space of n points on \mathbb{P}^1

Now let the positions of the n points $\{s_j\}$ vary on \mathbb{P}^1 , while fixing the $\{\mu_j\}$. We start with some basic definitions.

Definition 4. • Let \mathcal{P}_n denote the configuration space of n distinct ordered points on \mathbb{P}^1 . That is, $\mathcal{P}_n = (\mathbb{P}^1)^n \setminus \{z_i = z_j, i \neq j\}$.

- Let $\mathcal{P}_n^\Sigma := \mathcal{P}_n / \Sigma_n$, where Σ_n is the symmetric group on n letters which acts by permuting the s_j . This is the configuration space of n unordered points on \mathbb{P}^1 . It is an open subset of \mathbb{P}^n .
- We refer to $\pi_1(\mathcal{P}_n)$ as the braid group on n colored strings on \mathbb{P}^1 .

Similarly, $\pi_1(\mathcal{P}_n / \Sigma_n)$ is the braid group on n colorless strings on \mathbb{P}^1 .

The automorphism group of \mathbb{P}^1 is $PGL_2(\mathbb{C})$. An automorphism is completely determined by its action on any three distinct points of \mathbb{P}^1 . The diagonal action on $(\mathbb{P}^1)^n$ restricts to a free action on \mathcal{P}_n .

Definition 5. Let \mathcal{M}_n denote the moduli space of n distinct ordered points on \mathbb{P}^1 . It is the quotient of \mathcal{P}_n under the free diagonal action of $PGL_2(\mathbb{C})$. That is, $\mathcal{M}_n \equiv \mathcal{P}_n / PGL_2(\mathbb{C}) \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \{z_i = 0, 1, \infty, z_j, i \neq j\}$. Similarly, the moduli space of n distinct unordered points on \mathbb{P}^1 is the Σ_n quotient of \mathcal{M}_n , which we denote by \mathcal{M}_n^Σ .

Observe that $\mathcal{P}_n \cong \mathcal{M}_n \times PGL_2(\mathbb{C})$. Consequently, ignoring the choice of base point because the spaces in question are all connected, we have

Lemma 6.

$$\begin{aligned} \pi_1(\mathcal{P}_n) &= \pi_1(\mathcal{M}_n) \times \pi_1(PGL_2(\mathbb{C})) = \pi_1(\mathcal{M}_n) \times \mathbb{Z}/2\mathbb{Z} \\ \pi_1(\mathcal{P}_n^\Sigma) &= \pi_1(\mathcal{M}_n^\Sigma) \times \pi_1(PGL_2(\mathbb{C})) = \pi_1(\mathcal{M}_n^\Sigma) \times \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

As discussed in Section 2.1, local systems on \mathcal{P}_n are characterized up to isomorphism by representations of $\pi_1(\mathcal{P}_n, x_0)$ (the choice of x_0 is irrelevant up to isomorphism). The braid group acts naturally on the intersection homology of $L \rightarrow \mathbb{P}^1 \setminus S$, and thus defines a local system \mathcal{L} of rank $k - 2$ on \mathcal{P}_n . (See Section 2.5.2 for a detailed discussion of the braid representation.)

A point $p \in \mathcal{P}_n$ specifies a subset $S(p) \subset \mathbb{P}^1$ of n distinct points. Let $L_p \rightarrow \mathbb{P}^1 \setminus S(p)$ denote the DM local system defined by the data $(S(p), \mu)$. It turns out that what one would hope for is in fact true: namely, the vector spaces $IH_1(\mathbb{P}^1, L_p)$ arrange themselves over \mathcal{P}_n

into the local system $\mathcal{L} \rightarrow \mathcal{P}_n$. There is some ambiguity because L_p is not determined up to unique isomorphism by μ (Proposition 1). More precisely, one finds (adapting the arguments from cohomology to intersection cohomology is immediate when $\alpha_i \neq 1, \forall i$) [4, pp. 22, 26-27]:

Lemma 7. *Given μ there is a local system $\mathcal{L} \rightarrow \mathcal{P}_n$ with fiber at p given by $H^1(\mathbb{P}^1 \setminus S(p), L_p)$, where L_p is the DM local system with monodromy data μ . This local system is unique up to tensor product with a rank one local system $\mathcal{O} \rightarrow \mathcal{P}_n$.*

The ambiguity is removed by projectivizing the fibers. One may think of the resulting canonical flat projective bundle $P\mathcal{L}$ as one of a number of canonical Grassmanian bundles, constructed in the analogous way, on \mathcal{M}_n . More precisely, let $\dim_{\mathbb{C}}(IH_1(\mathbb{P}^1, L_p)) = k - 2$. For each $l, 1 < l < k - 2$, there is a canonical flat Grassmannian bundle $Gr(l, \mathcal{L})$, with fiber $Gr(l, k - 2)$ over each $m \in \mathcal{M}_n$.

Such bundles are characterized by their monodromy representation.

Definition 6. *Denote the monodromy group of \mathcal{L} by Γ , and the (canonical given μ) monodromy group associated to $P\mathcal{L}$ by $P\Gamma$.*

Furthermore, because the $PGL_2(\mathbb{C})$ action is trivial on the projective fibers, this flat bundle of projective spaces descends to \mathcal{M}_n . Alternatively, the projective representation of $\pi_1(\mathcal{P}_n)$ encoded in the projectivization of the local system \mathcal{L} is simultaneously a projective representation for $\pi_1(\mathcal{M}_n)$ by Lemma 6, and so canonically describes a flat projective space bundle on \mathcal{M}_n . To be more precise, let Θ denote the projective monodromy group of $P\mathcal{L} \rightarrow \mathcal{M}_n$. We verify that $P\Gamma = \Theta$.

Proposition 8. *The projective monodromy group Θ of the flat bundle of projective spaces $P\mathcal{L} \rightarrow \mathcal{M}_n$ is isomorphic to the projective monodromy group $P\Gamma$ of $P\mathcal{L} \rightarrow \mathcal{P}_n$.*

Proof. The flat bundle of projective spaces $P\mathcal{L} \rightarrow \mathcal{M}_n$ is isomorphic to a flat subbundle of the bundle of projective spaces $\mathbb{P}(IH_1(\mathbb{P}^1, L_p)) \rightarrow \mathcal{P}_n$: simply restrict the bundle via the inclusion $i: \mathcal{M}_n \subset \mathcal{P}_n, m = (m_0, \dots, m_{n-3}) \mapsto (0, 1, \infty, m_0, \dots, m_3)$. Thus the projective monodromy representation $\Theta(i_*\pi_1(\mathcal{M}_n))$ of $P\mathcal{L}$ is automatically a subgroup of $P\Gamma = \Theta(\pi_1(\mathcal{P}_n))$. By Lemma 6, $\pi_1(\mathcal{P}_n) = \pi_1(\mathcal{M}_n) \times \mathbb{Z}/2\mathbb{Z}$ so $\Theta(\pi_1(\mathcal{P}_n))$ is isomorphic to $\Theta(i_*(\pi_1(\mathcal{M}_n)))$ twisted by a character of $\mathbb{Z}/2\mathbb{Z}$. In particular, they define equivalent projective representations. \square

Remark 3. This justifies using $P\Gamma$ in either context, so henceforth we will not refer to Θ , only to $P\Gamma$. We freely use whichever interpretation is convenient, without further comment, throughout.

2.4 Definition of hypergeometric functions

In the preceding sections, all of the results were topological. Analysis enters via the definition of hypergeometric functions.

Proposition 9. *Let $L \rightarrow \mathbb{P}^1 \setminus S$ be a DM local system. There is an orthogonal “Hodge decomposition” $IH_1(\mathbb{P}^1, L) \cong IH_{1,0}(\mathbb{P}^1, L) \oplus IH_{0,1}(\mathbb{P}^1, L)$. The Hermitian form Ψ on $IH_1(\mathbb{P}^1, L)$ is positive definite on the subspace $IH_{1,0}(\mathbb{P}^1, L)$ and negative definite on $IH_{0,1}(\mathbb{P}^1, L)$.*

Proof. In general the decomposition follows from work of Saito [25, 26] when L is a local system of geometric origin in the sense of Grothendieck-Deligne (i.e., is a polarized variation of Hodge structure). It can be seen more directly by interpreting $IH_{1,0}$ as the space of holomorphic L -valued 1-forms and $IH_{0,1}$ as the space of holomorphic \bar{L} -valued 1-forms (i.e., anti-holomorphic L -valued 1-forms). See [4, Section 2] for details: the isomorphism of homology theories when $\alpha_s \neq 1$, from Lemma 3, together with the non-degeneracy of Ψ , carry over their argument unchanged. \square

Corollary 3.

$$\dim_{\mathbb{C}}(IH_{1,0}(\mathbb{P}^1, L)) = \left(\sum_i \mu_i\right) - 1$$

$$\dim_{\mathbb{C}}(IH_{0,1}(\mathbb{P}^1, L)) = \left(\sum_i 1 - \mu_i\right) - 1$$

Corollary 4. *If $\sum_i \mu_i = 2$ then Ψ is signature $(1, n - 3)$.*

Fix μ such that $\sum_i \mu_i = 2$. Pick coordinates on the fiber of $P\mathcal{L} \rightarrow \mathcal{M}_n$ at some point m_0 and extend by flatness. We use the fact that $IH_{1,0}(\mathbb{P}^1, L)$ is a distinguished 1-dimensional subspace, spanned by some ω_μ to define:

Definition 7. *The multi-valued holomorphic map $HG_\mu : \mathcal{M}_n \rightarrow \mathbb{P}^{k-2}$ is the coordinate expression of ω_μ . We call it the μ -hypergeometric function of Deligne-Mostow type.*

By construction, $HG_\mu(m)$ is an orbit of the projective monodromy group of $P\mathcal{L} \rightarrow \mathcal{M}_n$. HG_μ is completely determined up to automorphisms of \mathbb{P}^{k-2} .

Remark 4. Let Σ denote the symmetries of the list $\mu = (\mu_0, \dots, \mu_{n-1})$. Then Σ acts on \mathcal{M}_n as permutations of the associated coordinates s_i . It is clear from the definition that HG_μ descends to a map from \mathcal{M}_n^Σ . We denote this map by HG_μ as well. The domain will always be clear from context.

Remark 5. This definition of hypergeometric functions may be generalized to arbitrary μ by using the coordinate expression for $IH_{1,0}(\mathbb{P}^1, L)$ in the corresponding flat Grassmannian bundle. I am not aware of an analog of this definition in the literature.

Remark 6. When $\sum_i \mu_i = 2$ as above, there is in fact a unique holomorphic 1-form up to scaling. It may be written as:

$$\omega_\mu = \prod_i (z - s_i)^{-\mu_i} e \cdot dz,$$

where e is a horizontal multi-section of L (to cancel the monodromy of the function so that ω_μ is a well-defined section). This is the famous hypergeometric 1-form.

2.5 Uniformization by a complex ball

2.5.1 Complex ball and discrete subgroups of $PU(1, n)$

Let Ψ be an Hermitian form of signature $(1, n)$ on \mathbb{C}^{n+1} .

Definition 8. *The complex ball $\mathbb{B}^n \subset \mathbb{P}^n$ is defined to be the subset of points that lift to vectors in \mathbb{C}^{n+1} of strictly positive Ψ -length. In particular, Ψ defines a complex hyperbolic metric on \mathbb{B}^n .*

Remark 7. An Hermitian form over \mathbb{C} is determined up to equivalence (change of coordinates) by its signature. Consequently, \mathbb{B}^n is independent of Ψ . In particular, \mathbb{B}^n is a “ball” because, in an appropriate coordinate system $z = (z_0, \dots, z_n)$,

$$\begin{aligned} \Psi(z, z) &= |z_0|^2 - |z_1|^2 - \dots - |z_n|^2 > 0 \\ \Rightarrow 1 &> \left|\frac{z_1}{z_0}\right|^2 + \left|\frac{z_2}{z_0}\right|^2 + \dots + \left|\frac{z_n}{z_0}\right|^2 \end{aligned}$$

Let $PU(1, n)$ denote the group of projective linear transformations that lift to linear transformations on \mathbb{C}^{n+1} which preserve Ψ .

$$\begin{array}{ccccc} U(1, n) & \hookrightarrow & GL_{n+1}(\mathbb{C}) & = & Aut(\mathbb{C}^{n+1}) \\ \downarrow & & \downarrow & & \\ PU(1, n) & \hookrightarrow & PGL_{n+1}(\mathbb{C}) & = & Aut(\mathbb{P}^n) \end{array}$$

The complex ball has an interpretation as an Hermitian symmetric domain of type I .

Proposition 10. *The complex ball is a symmetric space.*

$$\mathbb{B}^n \cong PU(1, n)/P(U(1) \times U(n-1))$$

Furthermore $Aut(\mathbb{B}^n) \cong PU(1, n)$.

Proof. See [17], Volume II, Example 10.7, pages 282–285. \square

Proposition 11. *When $\sum \mu_i = 2$, the projective monodromy group $P\Gamma$ of $P\mathcal{L}$ is a subgroup of $PU(1, n-3)$ and so acts as automorphisms of the complex ball \mathbb{B}^{n-3} .*

Proof. The braid group acts via compactly supported isotopies on \mathbb{P}^1 . These isotopies induce automorphisms of the local system $L_{p_0} \rightarrow \mathbb{P}^1 \setminus S(p_0)$. Any such automorphism is multiplication in the fibers by \mathbb{C}^* (by Proposition 1). This in turn induces a constant \mathbb{C}^* rescaling of $IH_1(\mathbb{P}^1, L_{p_0})$, and so a real rescaling of the skew-Hermitian intersection pairing. Hence up to scaling the braid group action preserves Ψ . When $\sum \mu_i = 2$, the signature of Ψ is $(1, n-3)$ by Proposition 7. It follows that $P\Gamma \subset PU(1, n-3)$. And so the braid group acts through $P\Gamma$ as automorphisms of the complex ball \mathbb{B}^{n-3} . \square

Corollary 5. *Assume $\sum_i \mu_i = 2$ and let $|S| = n$. Then the multi-valued map $HG_\mu : \mathcal{M}_n \rightarrow \mathbb{B}^{n-3} \subset \mathbb{P}^{n-3}$.*

Proof. Because $\sum_i \mu_i = 2$, by Proposition 7 together with Proposition 9 one sees that $\dim_{\mathbb{C}}(IH_{1,0}(\mathbb{P}^1, L_m)) = 1$. Because $IH_{1,0}(\mathbb{P}^1, L_m)$ is positive definite, it follows that the point $\mathbb{P}(IH_{1,0}(\mathbb{P}^1, L_m))$ is an element of the ball $\mathbb{B}^{n-3} \subset \mathbb{P}(IH_1(\mathbb{P}^1, L_m)) = \mathbb{P}^{n-3}$. Recall $HG_\mu(m)$ is defined to be the $P\Gamma$ orbit of this point $\mathbb{P}(IH_{1,0}(\mathbb{P}^1, L_m)) \in \mathbb{P}^{n-3}$. By the Proposition, $P\Gamma$ acts as automorphisms of the ball, so $HG_\mu(m) \subset \mathbb{B}^{n-3}$. \square

Definition 9. *A discrete subgroup of a Lie group is an infinite subgroup for which the subspace topology is the discrete topology. A (group-theoretic) lattice is a co-finite volume discrete subgroup of a Lie group. We may sometimes informally refer to “discrete group” when we really mean a discrete subgroup.*

Example: Discrete subgroups like $SL(2, \mathbb{Z})$ and its congruence mod 2 subgroup $\Gamma(2)$ are lattices. One can check $\Gamma(2)$ arises as the monodromy group for the 4-point case, where $\mu_i = \frac{1}{2}, \forall i$.

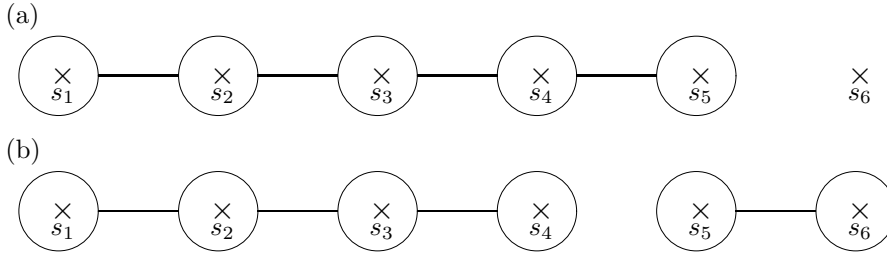
2.5.2 Monodromy: Braid Action on $IH_1(\mathbb{P}^1, L)$

The monodromy group Γ of $\mathcal{L} \rightarrow \mathcal{P}_n$ is the representation of the spherical n -strand braid group, $\pi_1(\mathcal{P}_n)$, on $IH_1(\mathbb{P}^1, L)$. As a general reference for standard results on the braid group that we use, see [16, Section 5] and the references contained therein.

Let $R_{i,i+1}$ denote the braid group “transposition” element that braids s_{i+1} about s_i and is the identity on $s_k, k \neq i+1$. It can be realized by a compactly supported isotopy of \mathbb{P}^1 that moves s_{i+1} along a counter-clockwise circle that encloses s_i and is the identity in neighborhoods of $s_k, k \neq i+1$. A well-known result is:

Lemma 8. *The spherical braid group on n strands is generated by the “transpositions” $R_{i,i+1}$ and $R_{n,1}$.*

Once a basis for $IH_1(\mathbb{P}^1, L)$ is chosen, then the action of these generators can be written in terms of explicit matrices. For simplicity we *assume* that all of the local monodromies are non-trivial, that is, $\mu_i \notin \mathbb{Z}, \forall i$. In Corollary 2 we constructed a basis for $IH_1(\mathbb{P}^1, L)$, taken from $\{I_{i,i+1} \equiv \frac{1}{\alpha_i-1}\hat{C}_i + \hat{\gamma} + \frac{1}{1-\alpha_{i+1}}\hat{C}_{i+1}\}_{i \in \{1, \dots, n-1\}} \cup \{I_{n,1}\}$. Roughly speaking, any $n-2$ cycles from this set form a basis. There are two possibilities: either (a) a point s_j is “isolated” or (b) some cycle $I_{i,i+1}$ (or $I_{n,1}$) is “isolated”.



Because all of the points are assumed to have non-trivial monodromy, the only way to violate the condition of Corollary 2 is with an “isolated” cycle $I_{i,i+1}$ for which $\mu_i + \mu_{i+1} \in \mathbb{Z}$. This will always be a counter-example to Corollary 2.

Counter-Example: Choose a partition (S_1, S_2) which does not satisfy the assumption of Corollary 2, so that $\sum_{i \in S_1} \mu_i \in \mathbb{Z}$. Then there exists a local system L_{S_1} on $\mathbb{P}^1 \setminus S_1$ defined by assigning μ_i to $s_i \in S_1$. Then the cycles $I_{i,i+1}$ cannot be linearly independent, because

$IH_1(\mathbb{P}^1, L_{S_1})$ is $(|S_1| - 2)$ -dimensional and there $(|S_1| - 1)$ cycles. A simple example is $|S| = 4$, $\mu_i = \frac{1}{2}, \forall s_i \in S$, with $S_1 = \{s_1, s_2\}$ and $S_2 = \{s_3, s_4\}$.

We will give partitions that do not exhibit this pathology a suggestive name.

Definition 10. A partition of S into subsets S_1 and S_2 where $\sum_{i \in S_1} \mu_i \notin \mathbb{Z}$ (or equivalently with S_2) stable partitions.

To study $R_{i,i+1}$ it is convenient to take advantage of the above flexibility in the choice of basis so as to “isolate” s_i and s_{i+1} , like in the above picture (b) of a “good basis.”

Lemma 9. Assume $S_1 = \{s_i, s_{i+1}\}, S_2 = S \setminus S_1$ defines a stable partition of S . In the good basis above, $R_{i,i+1}$ acts as the identity on the space spanned by the $n - 3$ remaining basis vectors. Furthermore, it acts as an order k complex rotation, for k the denominator of the fraction (in lowest terms) $\mu_i + \mu_{i+1}$, on the remaining basis vector $I_{i,i+1}$. More specifically, it acts on $I_{i,i+1}$ as multiplication by $e^{2\pi i(\mu_i + \mu_{i+1})}$.

Proof. Because (S_1, S_2) is a stable partition, these cycles form a basis. It is immediate that $R_{i,i+1}$ acts as the identity on the $n - 3$ intersection homology generators associated to S_2 , because the isotopy corresponding to the braid action is the identity away from a small compact set that contains s_i and s_{i+1} but no other s_k .

The action on $I_{i,i+1}$ is more involved. A formal argument can be adapted almost *mutatis mutandis* from [4, Proposition 9.2, pp.46-47]. Informally it is easy to see using a “relative position” argument. A counter-clockwise motion of s_{i+1} relative to a fixed s_i may be thought of as a counter-clockwise motion of s_i relative to a fixed s_{i+1} , with one full loop corresponding to one full loop. The section therefore is scaled by the local monodromy of each, namely $\alpha_1 \cdot \alpha_2 = e^{2\pi i(\mu_i + \mu_{i+1})}$. \square

Definition 11. A finite order complex linear transformation T with a hyperplane as its fixed point locus is called a complex reflection. The mirror of the reflection is the fixed hyperplane. If T preserves a hyperbolic Hermitian form Ψ , then we say T is a complex hyperbolic reflection.

Proposition 12. Assume the partition $S_1 = \{s_i, s_{i+1}\}, S_2 = S \setminus S_1$ is a stable partition of S . Then $R_{i,i+1}$ is a complex hyperbolic reflection of order k . The mirror of $R_{i,i+1}$ is the Ψ -orthogonal complement of the basis vector $I_{i,i+1}$.

Proof. By Lemma 9, $R_{i,i+1}$ is an order k complex reflection. By Proposition 11, Γ preserves the hyperbolic Hermitian form Ψ on $IH_1(\mathbb{P}^1, L)$. By Lemma 8 $R_{i,i+1}$ acts on $IH(\mathbb{P}^1, L)$ as a generator of Γ , and so it must preserve the hyperbolic structure.

The intersection pairing of $I_{i,i+1}$ with any of the remaining $n - 3$ basis vectors is trivial because their geometric supports do not intersect. These vectors associated to S_2 therefore span the Ψ -orthogonal complement of $I_{i,i+1}$. By Lemma 9, $R_{i,i+1}$ acts trivially on the S_2 basis vectors, and non-trivially on $I_{i,i+1}$. Hence the Ψ -orthogonal complement is the mirror of $R_{i,i+1}$. \square

Remark 8. If $S_1 = \{s_i, s_{i+1}\}$ does not define a stable partition, then observe that by Lemma 4, $I_{i,i+1}$ has Ψ -length zero, i.e., is isotropic.

For explicit computations it is useful to have the action of $R_{i,i+1}$ for all i with respect to a single fixed basis. This also makes the ring of integers $R = \mathbb{Z}(\zeta_d)$ over which Γ is defined transparent. Of course, R is the same as the base ring of the module-theoretic lattice $(IH_1(\mathbb{P}^1, L), \Psi)$, since Γ acts as a monodromy group.

Proposition 13. In the standard basis $I_{j,j+1}, j \in \{1, \dots, n - 2\}$, the reflection $R_{i,i+1}$ is a matrix with entries $R_{i,i+1}(a, b)$:

$$R_{i,i+1}(a, b) = \begin{cases} 1, & a = b \neq i \text{ and } |a - b| > 1 \\ \alpha_i \cdot \alpha_{i+1}, & a = b = i \\ 1 - \alpha_{i+1}, & a = i \text{ and } b = i - 1 \\ \alpha_{i+1}(1 - \alpha_i), & a = i \text{ and } b = i + 1 \\ 0, & \text{elsewhere.} \end{cases}$$

2.5.3 Uniformization: INT and ΣINT

To date we have considered the moduli space \mathcal{M}_n of n *distinct* points on \mathbb{P}^1 . Choosing μ is equivalent to choosing a line bundle on $(\mathbb{P}^1)^n$, and in fact uniquely determines a $SL_2(\mathbb{C})$ -linearization of the diagonal $SL_2(\mathbb{C})$ action. This means there is a well-defined compact GIT quotient, $\overline{\mathcal{M}}_{n,\mu}$. Let us denote the quasi-projective stable locus by $DM(n, \mu)$. The key insight that drives [4] is that HG_μ extends uniquely over $DM(n, \mu)$.

The main result of the paper of Deligne and Mostow [4] is that, for a finite list of μ , HG_μ has a single-valued inverse Φ_μ , and so the bottom map in the following diagram is an isomorphism of complex analytic spaces.

$$\begin{array}{ccc} & \Phi_\mu & \mathbb{B}^{n-3} \\ & \searrow^{HG_\mu} & \downarrow / \sim \\ \mathcal{M}_{n,\mu} & \xrightarrow{\cong} & \Gamma \backslash \mathbb{B}^{n-3} \end{array}$$

In fact they show more. For such μ , the uniformization extends, as an isomorphism of varieties, to the GIT compactification $\overline{\mathcal{M}}_{n,\mu}$ (including the semi-stable points) on the one hand and the Baily-Borel compactification $\overline{PT \backslash \mathbb{B}^{n-3}}^{BB}$ on the other. In short, “GIT = Baily-Borel”.

Their original sufficiency criterion for μ is simple to check.

Condition INT : Assume that the numbers μ_j defined by $\alpha_j = e^{2\pi i \mu_j}$, $0 < \mu_j < 1$ satisfy $\sum \mu_i = 2$. For all $s \neq t$ in S such that $\mu_s + \mu_t < 1$, require that $(1 - \mu_s - \mu_t)^{-1} \in \mathbb{Z}$.

Theorem 1 (INT [4]). *If Condition INT holds, then Γ is a lattice in the projective unitary group $PU(1, n-3)$. Moreover, $DM(n, \mu) \cong PT \backslash \mathbb{B}^{n-3}$, and indeed the isomorphism extends to their GIT and Baily-Borel compactifications as an isomorphism of varieties.*

The list of solutions is quite small, and in fact there is only one solution for $n = 7$ and none for $n > 7$. Furthermore, it would be nice to have a necessary and sufficient condition for Γ_μ to be discrete. In [23], Mostow develops a generalization of INT that largely fulfills that purpose.

Condition ΣINT : Assume that the numbers μ_j defined by $\alpha_j = e^{2\pi i \mu_j}$, $0 < \mu_j < 1$ satisfy $\sum \mu_i = 2$. Let S_1 be a subset of S with $\mu_s = \mu_t \forall s, t \in S_1$. For all $s \neq t \in S$ such that $\mu_s + \mu_t < 1$, require that

$$1 - \mu_s - \mu_t \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } s, t \in S_1 \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

Theorem 2 (ΣINT [23]). *If Condition ΣINT holds then Γ is a lattice in $PU(1, n-3)$. Let Σ denote the symmetric group of order $|S_1|$. Then $\mathcal{M}_n^\Sigma \cong PT_\Sigma \backslash \mathbb{B}^{n-3}$ for a group extension Γ_Σ of Γ by Σ , and furthermore this isomorphism extends to their GIT and Baily-Borel compactifications as an isomorphism of varieties.*

Remark 9. The “ Σ ” in ΣINT is meant to suggest the symmetric group. In essence, the idea behind ΣINT is to exploit repeated values in the list $\{\mu_j\}$ by constructing a uniformization for $DM(n, \mu)/\Sigma$. So the arguments in the proof largely reduce to the same arguments used for condition INT .

Remark 10. Whenever an example satisfies ΣINT , unless otherwise noted, we by default work with the quotient \mathcal{M}_n^Σ .

3 Eisenstein and Gaussian Ancestral Examples

3.1 Automorphisms of lattices

Let (L, Ψ) be a lattice over R , M a sublattice, and N the Ψ -orthogonal complement of M in L . Any automorphism of L restricts to an automorphism u_M of M and an automorphism u_N of N . Conversely, when do the automorphisms of M extend to automorphisms of L ?

To address the question we recall some basic ideas from the theory of lattices. We assume throughout that Ψ is non-degenerate, which is automatically true for Ψ as defined in Section 2.

Let L^* denote the dual lattice $\text{Hom}_R(L, R)$. The form Ψ induces a map $a_L : L \rightarrow L^*$, given by $x \mapsto \Psi(\cdot, x)$. Since Ψ is nondegenerate, a_L embeds L as a sublattice of L^* of finite index. (Drawing L^* as the usual “square” Cartesian lattice, one sees the sublattice $a_L(L)$ is the standard “pictorial” representation of the lattice L .) Many of the differences with the theory of vector spaces, where $V^* \cong V$, are captured by the discrepancy between L^* and $a_L(L)$.

Let $C(L) := L^*/a_L(L)$, and observe that it is a finite R -module. Furthermore, Ψ determines the Hermitian form Ψ^* on L^* , but now this form is valued in the field of fractions of R , denoted $F(R)$. This in turn induces a Hermitian form $\Psi_{C(L)}^*$ on the finite R -module $C(L)$. Note that $C(L)$ is not a “lattice” *per se*, because the form is valued in the group-theoretic quotient $F(R)/R$. Nonetheless, it is clear that any unitary transformation of L induces an automorphism of $C(L)$ that preserves $\Psi_{C(L)}^*$.

Assume now that (Λ, Ψ) is a *unimodular* R -lattice. To address the question above, consider a *primitive* sublattice M and its Ψ -orthogonal complement N . It turns out that there is a natural isomorphism $\alpha : C(M) \rightarrow C(N)$ that changes the sign of the forms $\Psi_{C(M)}^*$ and $\Psi_{C(N)}^*$ but otherwise preserves them. One can then see:

Proposition 14. [16, Appendix, pp.43-44] *Let M be a primitive sublattice of a unimodular lattice L . A pair of unitary transformations u_M of M and u_N of N , defining a unitary transformation (u_M, u_N) of $M \perp N$, is an automorphism of L if and only if the following diagram commutes:*

$$\begin{array}{ccc} C(M) & \xrightarrow{\alpha} & C(N) \\ u_M \downarrow & & \downarrow u_N \\ C(M) & \xrightarrow{\alpha} & C(N) \end{array}$$

There are two rings that principally concern us: the Gaussian and Eisenstein rings of integers.

Definition 12. *The ring \mathcal{G} of Gaussian integers is $\mathbb{Z}[\iota]$, where $\iota = \sqrt{-1}$. The ring \mathcal{E} of Eisenstein integers is $\mathbb{Z}[\omega]$, where ω is a primitive third root of unity.*

Remarkably, for \mathcal{G} and \mathcal{E} we don’t need an explicit description of α . The results follow from the properties that α is an isomorphism and (up to sign) preserves the form.

Corollary 6. *Let (Λ, Ψ) be a lattice over the ring $R = \mathcal{G}$ or \mathcal{E} . Let $z \in \Lambda$ be a primitive vector (i.e., z generates a primitive sublattice Rz in M) not of unit length. Let $\Lambda_0 \subset \Lambda$ be the sublattice that is Ψ -orthogonal to z . Then the map from the $\text{Aut}(\Lambda)$ -stabilizer of Λ_0 to $\text{Aut}(\Lambda_0)$ is an isomorphism. In particular, any automorphism of Λ_0 extends uniquely to an automorphism of Λ . If z is of unit length then the ambiguity in the extension is just the automorphism group of Rz , namely $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ for \mathcal{G} and \mathcal{E} respectively.*

Proof. The proofs in the Gaussian and Eisenstein cases are analogous. In each case the essential point is that $C(Rz)$ is isomorphic to $R/(r)$ for some $r \in R$. The unitary automorphisms (those preserving $\Psi_{C(Rz)}^*$) of $R/(r)$ are easily seen to be one of these: trivial (if r is a unit), $\mathbb{Z}/6\mathbb{Z}$ if R is Eisenstein and r not a unit, or $\mathbb{Z}/4\mathbb{Z}$ if R is Gaussian and r not a unit. Consider a unitary transformation u_M of Λ_0 . This induces a unitary transformation of $C(\Lambda_0)$, which, because it is isomorphic via α to $C(Rz)$, must be an element of the trivial group, $\mathbb{Z}/6\mathbb{Z}$, or $\mathbb{Z}/4\mathbb{Z}$ according to the cases above. Now, α itself acts (accounting for the sign change) as a unitary automorphism, so it satisfies the same trichotomy. In particular, one can always find an automorphism of $C(Rz)$ to “undo” α and so make the diagram from the Proposition commute. The only potential obstruction is that the requisite automorphism of $C(Rz)$ may not come from an automorphism u_N of Rz . But the unitary transformations of Rz are $\mathbb{Z}/6\mathbb{Z}$ for R Eisenstein or $\mathbb{Z}/4\mathbb{Z}$ for R Gaussian, so in fact one can always find such a u_N , and in particular, as long as r is not a unit, that u_N is determined uniquely by u_M . In other words, any automorphism of Λ_0 extends uniquely to an automorphism of Λ . If r is a unit, then the ambiguity is precisely the group of units in \mathcal{G} or \mathcal{E} . \square

In general, given a locally symmetric space (here, a ball quotient), it can be quite difficult to identify locally symmetric subspaces (here, subball quotients).

Definition 13. Let $P\Gamma$ be a discrete subgroup of $PU(1, n)$. Let \mathbb{B}^k be a subball of \mathbb{B}^n . In particular, \mathbb{B}^k is cut out by a (projective) linear constraint on the ambient \mathbb{P}^n . Let $P\Gamma_{Stab}$ denote the subgroup of $P\Gamma$ that preserves \mathbb{B}^k . Consider the image of \mathbb{B}^k in the ball quotient $P\Gamma \backslash \mathbb{B}^n$. We say that this image is a subball quotient if the map factors through an inclusion of $P\Gamma_{Stab} \backslash \mathbb{B}^k$ in $P\Gamma \backslash \mathbb{B}^n$.

$$\begin{array}{ccc} \mathbb{B}^k & \xrightarrow{\quad} & \mathbb{B}^n \\ \downarrow & & \downarrow \\ P\Gamma_{Stab} \backslash \mathbb{B}^k & \longrightarrow & P\Gamma \backslash \mathbb{B}^n \end{array}$$

The above Corollary tells us that, for a unimodular lattice over the Gaussian or Eisenstein integers, “arithmetically-defined” hyperballs \mathbb{B}^{n-1} correspond to codimension 1 subball quotients. Induction yields:

Corollary 7. For Λ a unimodular lattice of hyperbolic signature over $R = \mathcal{G}$ or \mathcal{E} , any primitive hyperbolic sublattice Λ_0 defines a subball quotient:

$$PAut(\Lambda_0) \backslash \mathbb{B}^{n-1} \subset PAut(\Lambda) \backslash \mathbb{B}^n$$

Proof. When Λ_0 is the Ψ -orthogonal complement of a primitive vector in Λ and is of hyperbolic signature this is a restatement of the previous Corollary. It is clear that the intersection of subball quotients is again a subball quotient. So, because Λ_0 is the Ψ -orthogonal complement of some primitive lattice, the statement follows by induction. \square

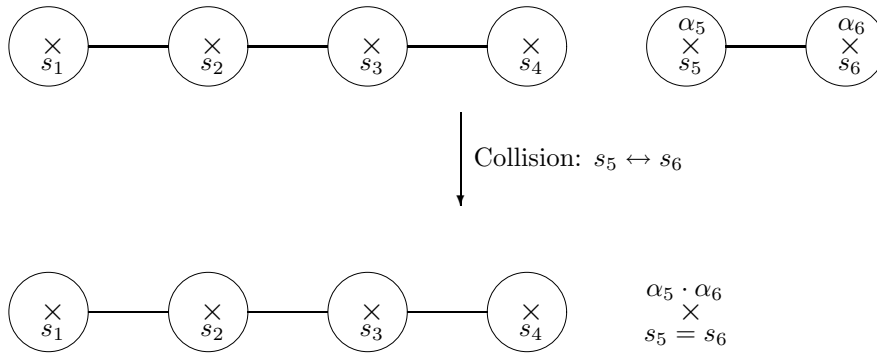
3.2 Organizing principle: Descendants by collision

All the Γ discussed in this section are assumed to satisfy ΣINT (and so in particular are group-theoretic lattices, i.e., discrete subgroups of $PU(1, n-3)$, defined over some ring of integers), thus $DM(n, \mu) \cong P\Gamma \backslash \mathbb{B}^{n-3}$. A collision between two points s_i and s_j is identified with the complement of a lattice vector, yielding a codimension 1 subball quotient in $P\Gamma \backslash \mathbb{B}^{n-3}$. This is implicit in Deligne and Mostow’s main theorems, as it is a part of the extension of the uniformization over the stable boundary of $DM(n, \mu)$. To be explicit, using the notation introduced in Section 2.5.2:

Lemma 10. Assume $\{s_i, s_j\} \cup S \setminus \{s_i, s_j\}$ is a stable partition. Let $S_{i,j}$ denote the sublocus consisting of all configurations of points for which s_i and s_j share a coordinate (i.e., have “collided”). The image of the principal branch of HG_μ restricted to $S_{i,j}$ is the mirror of $R_{i,j}$. Equivalently it is the Ψ -orthogonal complement of the vector in \mathbb{B}^{n-3} assigned to $I_{i,j}$.

Proof. Because it is a stable collision, $S_{i,j}$ is a nonempty subset of $DM(n, \mu)$. By Theorems 1 and 2, HG_μ is well-defined on $S_{i,j}$. For convenience, relabel the points to be s_i and s_{i+1} . $HG_\mu(m)$ is valued in $\mathbb{P}(IH_1(\mathbb{P}^1, L_m))$. Consider the good basis that “isolates” $I_{i,i+1}$, which exists by Corollary 2 because this is a stable partition of S . $HG_\mu(m)$ is (the projective image of) a linear combination of these basis vectors, or equivalently by Proposition 12, of $I_{i,i+1}$ and the basis vectors in its Ψ -orthogonal complement. When $s_i(m) = s_{i+1}(m)$ via a path 0 to m that does not cross a branch, a good basis for $IH_1(\mathbb{P}^1, L_m)$ is precisely (the flat translate of) the basis for the Ψ -orthogonal complement of $I_{i,i+1}$, denoted $I_{i,i+1}^\perp$; that is, $IH_1(\mathbb{P}^1, L_m)$ is the mirror of the complex reflection $R_{i,i+1}$.

In particular, for such m , $IH_{1,0}(\mathbb{P}^1, L) \subset I_{i,i+1}^\perp$, or equivalently, $HG_\mu(m) \in \mathbb{P}(I_{i,i+1}^\perp) \cap \mathbb{B}^{n-3}$. Because HG_μ is by assumption a uniformization, the image of $S_{i,i+1}$ is an open subset of the hyperball $\mathbb{I}_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}^\perp \subset \mathbb{B}^{n-3}$. But the sublocus is also a closed subset, so again because HG_μ is an isomorphism with $P\Gamma \backslash \mathbb{B}^{n-3}$, it must map to a closed set and hence the image is both open and closed in the hyperball, and so is the full hyperball. \square



Remark 11. By induction, a general collision sublocus is just the intersection of mirrors, and so the orthogonal complement of a collection of vectors.

There are precisely four Deligne-Mostow lattices that are generated by μ with all the μ_i equal-valued. These are the equally weighted $n = 4, 5, 6, 8$, and 12 point examples. For $n = 6$ and 12, R is the Eisenstein ring $\mathbb{Z}[\omega]$ (here ω is a primitive sixth root of unity), whereas for $n = 4$ and 8, R is the Gaussian ring $\mathbb{Z}[i]$.

One special feature of these lattices is that Γ_Σ is the *full* automorphism group of the associated module-theoretic lattice, rather than just a subgroup. In these two cases results can be found, from a different perspective and in different language, in the recent literature.

Theorem 3 ([21]). *The uniformizing group Γ_G for the Gaussian ancestral example, that is, the Deligne-Mostow example for 8 equally weighted ($\mu_i = \frac{1}{4}$) points, equals the full automorphism group of the corresponding lattice. That is, $\Gamma_G = \text{Aut}(\mathbb{Z}[i], \Psi_G)$.*

Theorem 4 ([1]). *The uniformizing group Γ_E for the Eisenstein ancestral example, that is, for 12 equally weighted ($\mu_i = \frac{1}{6}$) points, equals the full automorphism group of the corresponding lattice. That is, $\Gamma_E = \text{Aut}(\mathbb{Z}[\omega], \Psi_E)$.*

Definition 14. *We call these the ancestral Deligne-Mostow lattices. The 8 point case we call the Gaussian ancestral lattice and similarly the 12 point case we call the Eisenstein ancestral lattice. By a descendant lattice, we mean the subgroup of an ancestral lattice which is the stabilizer subgroup for a subconfiguration space (collision sublocus).*

Theorem 5. *Descendants of the ancestral lattices are themselves automorphism groups of the (module-theoretic) sub-lattices. The collision loci in $DM(1^8)$ and $DM(1^{12})$ are orbifold subball quotients.*

Proof. It suffices to check for codimension 1, the rest follow by induction. By Lemma 10 the image under (a branch of) HG_μ of a stable collision of a pair of points is the Ψ -orthogonal complement of a vector $I_{i,j}$. The vector lies on the lattice Λ in $IH_1(\mathbb{P}^1, L)$, so the complement defines a sublattice Λ_0 . By Lemma 4, $I_{i,j}$ has negative length, so its Ψ -orthogonal complement is hyperbolic. By Corollary 6, any automorphism of Λ_0 therefore extends to an automorphism of Λ . So the stabilizer subgroups are in fact themselves automorphism groups of sub-lattices. The non-uniqueness of the extension is the order of $R_{i,j}$ as a complex reflection, which is non-trivial by Lemma 9, so these are orbifold loci. \square

Remark 12. Making use of the three common meanings of “lattice” in mathematics — poset, group theoretic, and module theoretic — this Theorem tells us we have described, amusingly, a “lattice of lattices which are automorphisms of lattices”.

It is straightforward to observe that the equally weighted $n = 6$ case, defined by $\mu = (\frac{1}{3}, \dots, \frac{1}{3})$, is a descendant of the Eisenstein example, where the 12 points have all collided in pairs. Since the $n = 5$ case is two complex dimensional, it has no descendants of Deligne-Mostow type (and precisely one descendant of dimension 1).

Corollary 8. *The only equally weighted examples with proper Deligne-Mostow descendants are the Gaussian and Eisenstein ancestral examples.*

Corollary 9. *For $n > 7$, all the Deligne-Mostow lattices are (finite index sublattices of) descendants of the Eisenstein and Gaussian ancestral lattices. Similarly, all but one of the $n = 7$ examples is a descendant, and a number of the remaining ones ($n = 5, 6$) are as well*

(see Chart). Furthermore, the original list due to Mostow is in error, for it misses a number of descendant solutions.

Proof. This follows by direct observation and comparison with Mostow's chart [23]. \square

Remark 13. Thurston, working on the problem of enumerating flat metrics with cone singularities on S^2 , corrected Mostow's computations by a computer check, and his list should be complete [27].

Remark 14. We show in [8] that the moduli space of cubic surfaces inherits a ball quotient structure, agreeing with that discovered in [6], from the Eisenstein descendant $DM(2^5, 1^2)$, which is one of the examples missed by Mostow's tables.

4 Pull-back Construction

4.1 Intersection homology under pull-back

Fix a finite subset $T \subset \mathbb{P}^1$ and define a rank 1 Deligne-Mostow local system l_T on $\mathbb{P}^1 \setminus T$ with monodromy ν and ring of definition R . Consider a map $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Denote the inverse image sheaf, known henceforth as the *pull-back local system*, on $\mathbb{P}^1 \setminus \pi^{-1}(T)$ by π^*l_T . Because it is rank 1, π^*l_T is determined by local monodromies at the elements of $\pi^{-1}(T)$ (by Proposition 1), which in turn can be expressed in terms of ν and the ramification indices of π . More precisely:

Lemma 11. *Let $p_{i,j}$ denote the points of the set $\pi^{-1}(t_j)$ and let $r_{i,j}$ denote the ramification index of π at $p_{i,j}$. Then π^*l_T is the Deligne-Mostow local system on $\mathbb{P}^1 \setminus \pi^{-1}(T)$ defined by the local monodromy data $r_{i,j} \cdot \nu_j$ at $p_{i,j}$. It contains the pull-back local subsystem $\pi^*l_T(R)$ with fiber R .*

We now study how π induces maps on intersection homology. One approach is, using the formalism due to Deligne developed in [14], to define π_* and its adjoint map π^* at the level of the intersection chain complexes for the cover X and the base Y . To avoid introducing new notation, it is more direct to follow [15], and use the following definition.

Definition 15. *A subanalytic map $f : X \rightarrow Y$ between two subanalytic pseudo-manifolds is called placid if there exists a subanalytic stratification of Y such that for each stratum S in Y we have*

$$\text{codim}_X f^{-1}(S) \geq \text{codim}_Y(S)$$

Any branched covering is placid, so in particular π is placid, where the strata for Y are given by $(T, \mathbb{P}^1 \setminus T)$ and those for X by $(\pi^{-1}(T), \mathbb{P}^1 \setminus \pi^{-1}(T))$.

Intersection homology is a bivariant functor for placid maps, where the contravariant induced map may shift degrees. Although the following Proposition is proven in [15, Proposition 4.1] for intersection homology valued in the trivial rank 1 rational local system (i.e., the constant sheaf with stalk \mathbb{Q}), its proof immediately generalizes to intersection homology valued in a rank 1 local system $L \rightarrow Y$ and in the pull-back local system $f^*L \rightarrow X$. (Alternatively, one can prove this formally, for the topological definition of placid maps, using Deligne's construction of intersection homology.)

Proposition 15. *Suppose $f : X \rightarrow Y$ is a placid map. Let $L \rightarrow Y$ be a rank 1 local system, and let f^*L denote the pull-back local system on X . Then pushforward of chains and pull-back of generic chains induces homomorphisms on intersection homology,*

$$\begin{aligned} f_* : IH_i(X, f^*L) &\rightarrow IH_i(Y, L) \\ f^* : IH_i(Y, L) &\rightarrow IH_{i+\dim(X)-\dim(Y)}(X, f^*L) . \end{aligned}$$

In particular, $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ induces a map $\pi^ : IH_1(\mathbb{P}^1, l_T) \rightarrow IH_1(\mathbb{P}^1, \pi^*l_T)$. This map respects the intersection pairing.*

Remark 15. Indeed, one does not need the fiber to be a field; as is remarked in [15] after the proof of the Proposition, the same argument carries over for any coefficient ring R . The same result thus holds for any local subsystems $L(R)$, with R a subring of \mathbb{C} . In that event, π^* is a map of R -modules.

Furthermore, (using the differential form model for intersection cohomology) pulling back a (anti-)holomorphic form via an algebraic map yields a (anti-)holomorphic form, so the orthogonal decomposition into $IH^{1,0} \oplus IH^{0,1}$ is respected. The isomorphism with intersection homology via the intersection pairing, $IH^1(\mathbb{P}^1, L) \rightarrow IH_1(\mathbb{P}^1, L), z \mapsto (\cdot, z)$, tautologically respects the orthogonal decomposition. Thus the map π^* on intersection homology also respects the Hodge decomposition.

Proposition 16. *The map $\pi^* : IH_1(\mathbb{P}^1, l_T) \rightarrow IH_1(\mathbb{P}^1, \pi^* l_T)$ preserves the intersection pairing and hence the Hermitian form Ψ , in the sense $\Psi(\alpha, \beta) = \Psi(\pi^*(\alpha), \pi^*(\beta))$. In addition π^* respects the orthogonal direct sum (Hodge) decomposition, so that the subspace $\pi^*(IH_{1,0}(\mathbb{P}^1, l_T)) \subset IH_{1,0}(\mathbb{P}^1, \pi^* l_T)$ and $\pi^*(IH_{0,1}(\mathbb{P}^1, l_T)) \subset IH_{0,1}(\mathbb{P}^1, \pi^* l_T)$.*

Remark 16. This follows more formally from work of Saito on mixed Hodge modules. See [25]. Furthermore, as a consequence, one can amplify Remark 15. If one works with local subsystems whose fibers are the ring $R \subset \mathbb{C}^*$, then π^* is a map of Hermitian lattices over R .

4.2 Hurwitz spaces and \mathcal{S}_π

Now we vary π while preserving the ramification behavior over the fixed branch locus T . For any π in this family, the pull-back local system $\pi^* l_T$ will have the same monodromy data μ . As π varies, the coordinates of the points of $\pi^{-1}(T)$ vary.

Definition 16. *Let $S \subset \pi^{-1}(T)$ denote the subset of points with nontrivial local monodromy in $\pi^* l_T$.*

In particular S varies with π ; we write this dependence as $S(\pi)$. Let us be more precise:

Definition 17. *By the T -ramification class of π , denoted \mathcal{H}_π , we mean the subset of all maps $\pi' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ satisfying three conditions.*

1. π' has the same degree as π .
2. The ramification indices $r_{i,j}$ over points $t_j \in T$ are the same for π' and π .
3. π' is in the same connected component as π (with respect to the subspace topology of the standard topology on the space of maps between compact sets).

The notation is meant to emphasize the link with Hurwitz spaces, i.e., spaces of curve covers up to equivalence, since these self-maps of \mathbb{P}^1 are curve covers with constrained ramification. Equivalence of curve covers is given by the $PGL_2(\mathbb{C})$ action on $\mathbb{P}^1 = \mathbb{P}(V)$, which lifts to an $SL_2(\mathbb{C})$ action on V . Of course, $SL_2(\mathbb{C})$ also acts on the sets $\pi^{-1}(T)$ and S via $Sym^{|\pi^{-1}(T)|}(V)$ and $Sym^{|S|}(V)$ respectively. We denote the induced $SL_2(\mathbb{C})$ -equivariant algebraic maps by

$$\begin{aligned} \mathcal{H}_\pi &\xrightarrow{i_T} \mathbb{P}^{|\pi^{-1}(T)|} \xrightarrow{p_S} \mathbb{P}^{|S|} \\ \pi' &\mapsto \pi'^{-1}(T) \mapsto S(\pi'). \end{aligned}$$

We are interested in the space of all configurations $S(\pi') \subset \mathbb{P}^1$ where $\pi' \in \mathcal{H}_\pi$.

Definition 18. *Let \mathcal{S}_π denote the $SL_2(\mathbb{C})$ -quotient of the image subvariety $p_S \circ \iota_T(\mathcal{H}_\pi) \subset \mathbb{P}^{|S|}$.*

A curve cover $\pi \in \mathcal{H}_\pi$ is simply a rational function on \mathbb{P}^1 . Let $V \cong \mathbb{C}^2$ with coordinates (u, v) . The set of all degree d maps $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is given by:

$$\left\{ \frac{N(u, v)}{D(u, v)} \mid N, D \in Sym^d(V) \right\} = \mathbb{P}(Sym^d(V) \oplus Sym^d(V)).$$

In particular \mathcal{H}_π is an $SL_2(\mathbb{C})$ -invariant subvariety of $\mathbb{P}(Sym^d(V) \oplus Sym^d(V))$.

Proposition 17. *ι_T is injective*

Proof. Observe that the numerator $N(u, v)$ determines $\pi^{-1}(0)$ and the denominator $D(u, v)$ determines $\pi^{-1}(\infty)$. Conversely these two sets of points determine π up to scaling.

Since $|T| \geq 3$, use an automorphism of \mathbb{P}^1 to assign $t_0 = 0$, $t_1 = 1$, and $t_2 = \infty$. Then $\pi^{-1}(t_0)$ and $\pi^{-1}(t_2)$ determine π up to scaling. But if $\pi' = \lambda\pi$, then $\pi'(\pi^{-1}(t_1)) = \lambda$, so in fact $\pi^{-1}(t_1)$ determines the scaling factor. \square

The dimension of \mathcal{H}_π is easy to compute, as it is essentially an application of Riemann-Hurwitz.

Proposition 18. *When it exists, the T -ramification class \mathcal{H}_π , where π is degree d , is a $SL_2(\mathbb{C})$ -invariant subvariety of $\mathbb{P}(\text{Sym}^d(V) \oplus \text{Sym}^d(V))$, with codimension equal to $\sum_{i,j} (r_{i,j} - 1)$.*

Proof. The Riemann-Hurwitz formula here states $2(d - 1) = \sum_k (r_k - 1)$ where the sum is over *all* ramification points. Up to $SL_2(\mathbb{C})$ -equivalence, generically the set of covers is in one-to-one correspondence with the set of coordinates of the ramification points. The requirement that a ramification point with index $r_{i,j}$ map to a specific $t_j \in \mathbb{P}^1$ is therefore a codimension $r_{i,j} - 1$ condition, and all of these are independent. \square

Remark 17. Equivalently, the dimension, accounting for $SL_2(\mathbb{C})$ -equivalence, is the number of “free” or “excess” simple (order 2) ramification points (i.e., those not in $\pi^{-1}(T)$) allowed by Riemann-Hurwitz.

4.3 Restricting hypergeometric functions: \mathcal{S}_π and subball quotients

Let $n = |S|$, and let the monodromy data for S be μ . If (S, μ) can be realized via pullback by π , then $\mathcal{S}_\pi \subset DM(n, \mu)$. We consider the multi-valued hypergeometric function HG_μ , defined on $DM(n, \mu)$, restricted to \mathcal{S}_π . The restricted hypergeometric function satisfies a linear constraint. More precisely:

Lemma 12. $\omega_\mu(\mathcal{S}_\pi)$ is Ψ -orthogonal to the well-defined marked subspace $\pi^*(IH_{0,1}(\mathbb{P}^1, l_T))$. In particular, a branch of $HG_\mu(\mathcal{S}_\pi)$ lies in a subball $\mathbb{B}^k \subset \mathbb{P}((\pi^*(IH_{0,1}(\mathbb{P}^1, l_T)))^\perp)$.

Proof. Let R be the ring of integers in $\mathbb{Q}(\zeta_d)$. The restriction of $\mathcal{L}_\mu \rightarrow \mathcal{M}_{|S|}$ to \mathcal{S}_π is a local system with a marked local subsystem $\pi^*(IH_{0,1}(\mathbb{P}^1, l_T(R)))$. The fibers are of the subsystem are:

$$\pi^*(IH_{0,1}(\mathbb{P}^1, l_T(R))) \subset IH_1(\mathbb{P}^1, \pi^*l_T(R)) \cap IH_{0,1}(\mathbb{P}^1, \pi^*l_T).$$

The marked subspace $\pi^*(IH_{0,1}(\mathbb{P}^1, l_T))$ is independent of choice of π in \mathcal{S}_π , because \mathcal{S}_π is connected and because $\pi^*(IH_1(\mathbb{P}^1, l_T(R)))$ is a sublattice and so is invariant under continuous deformations of π . Then, for any $\pi \in \mathcal{S}_\pi$, $\omega_\mu(S(\pi)) \in (\pi^*(IH_{0,1}(\mathbb{P}^1, l_T)))^\perp$ by Proposition 16. Choosing consistent coordinates by not extending over branch loci, the rest follows immediately from the definition of HG_μ . \square

So a branch of the restricted hypergeometric function always lies in a subball. We use this fact, applied to Deligne-Mostow uniformizations of Eisenstein or Gaussian type, to produce \mathcal{S}_π that give subball quotients. Let $\overline{\mathcal{S}_\pi}$ denote the closure of \mathcal{S}_π in $DM(n, \mu)$.

Theorem 6. *Let μ be of Eisenstein or Gaussian type satisfying ΣINT . If $\dim_{\mathbb{C}}(\mathcal{S}_\pi) = \dim_{\mathbb{C}}(\pi^*(IH_{0,1}(\mathbb{P}^1, l_T(R))))$, then $\overline{\mathcal{S}_\pi} \subset DM(n, \mu)$ is the Baily-Borel compactification of the subball quotient $\Gamma_{Stab} \backslash \mathbb{B}((\pi^*(IH_{0,1}(\mathbb{P}^1, l_T)))^\perp)$.*

Proof. We know $\overline{\mathcal{S}_\pi}$ is an algebraic subvariety of $\mathcal{M}_{n,\mu}$. Furthermore, the subball quotient $\overline{P\Gamma_{Stab} \backslash \mathbb{B}^k}^{BB}$ is an irreducible algebraic subvariety of $\overline{P\Gamma \backslash \mathbb{B}^{n-3}}$. Because Φ is an isomorphism, and $\Phi(\mathcal{S}_\pi) \subset P\Gamma_{Stab} \backslash \mathbb{B}((\pi^*(IH_1(\mathbb{P}^1, l_T)))^\perp)$, we see \mathcal{S}_π and its closure inject as subvarieties. Since the only equal-dimensional closed subvariety of an irreducible variety is the irreducible variety itself, as long as the dimensions are equal one concludes Φ restricts to give an isomorphism of these two varieties. \square

It is therefore important to compute the dimension of \mathcal{S}_π . We know it equals the dimension of \mathcal{H}_π precisely when the map p_S restricted to \mathcal{H}_π is generically finite-to-one. One interesting class of examples is when all the non-trivial monodromy lies over a single point of T .

Lemma 13. *Assume $S = \pi^{-1}(t_j)$ for some $t_j \in T$. If, for some k , t_k has local monodromy $\nu_k = \frac{n_k}{d_k}$ such that $d_k > 2$, then $\dim_{\mathbb{C}}(\mathcal{S}_\pi) = \dim_{\mathbb{C}}(\mathcal{H}_\pi)$.*

Proof. Using the $SL_2(\mathbb{C})$ -action one may equivalently assume $S = \pi^{-1}(\infty)$, that $t_k = 0$, and that some $t_i = 1$. We claim there are only finitely many π' in $p_S^{-1}(S)$. For $\pi'^{-1}(0)$ to consist of points with trivial local monodromy, each point must be ramified to order a multiple of at least 3 which is a codimension at least $\frac{d}{3}(3-1)$ condition. Similarly and independently, over 1 every point is ramified to order a multiple of at least 2, which is codimension at least $\frac{d}{2}(2-1)$. Each fiber of p_S must therefore be less than d dimensional. But the denominator of π is determined up to scaling by S , which is a d -dimensional condition. So the generic fiber of p_S is finite-to-one. \square

4.4 Key examples: $|T| = 3$ and moduli spaces of inhomogeneous binary forms

We now explicitly work out the simplest examples. Assume $|T| = 3$, $\Sigma\mu_i = 2$, $\Sigma\nu_i = 1$, μ satisfies ΣINT , and μ is Eisenstein or Gaussian. Specializing our previous results, we obtain:

Corollary 10. $\overline{\mathcal{S}_\pi} \subset DM(n, \mu)$ is a subball quotient if and only if π^* is non-trivial and \mathcal{S}_π is codimension 1.

Proof. Here $IH_1(\mathbb{P}^1 \setminus T, l_T)$ is one-dimensional and purely anti-holomorphic. The image under pull-back is either trivial or one-dimensional, and purely anti-holomorphic. \square

It is worthwhile to completely classify the solutions in a special case. By Corollary 10, we need to compute $\dim_{\mathbb{C}}(\mathcal{S}_\pi)$. Lemma 13 suggest we consider $S = \pi^{-1}(t_i)$. Restrict further to the case where all points over a given $t_j \in T$ have the same ramification index. We think of this as a weak form of a “Galois” condition on π , and so define:

Definition 19. The pair (π, T) possesses property G if, for $t_j \in T$, all the points in $\pi^{-1}(t_j)$ have the same ramification index r_{t_j} .

Up to automorphisms of \mathbb{P}^1 , we may take $T = \{0, 1, \infty\}$.

Proposition 19. Let $T = \{0, 1, \infty\}$ and assume π is degree d . Then π has property G , with $r_0 = a$, and $r_1 = b$ if and only if

$$\pi = \frac{A^a(u, v)}{A^a(u, v) + B^b(u, v)}, \quad \deg(A) = \frac{d}{a}, \quad \deg(B) = \frac{d}{b},$$

where neither $A(u, v)$ nor $B(u, v)$ have repeated roots, and where $A^a(u, v) + B^b(u, v)$ either has no repeated roots or is of the form $C(u, v)^c$ for $c|d$ where $C(u, v)$ has no repeated roots.

Proof. The points in $\pi^{-1}(x)$ are the solutions to $\pi(u, v) = x$. The ramification index of a point is its multiplicity as a solution. Write $\pi = N(u, v)/D(u, v)$. The numerator $N(u, v)$ and the denominator $D(u, v)$ are both degree d homogeneous polynomials.

Let π satisfy the assumptions. The points of $\pi^{-1}(0)$ are simply the roots of $N(u, v)$. Property G says they all must have the same multiplicity, and so there are $k_1 = \frac{d}{a}$ distinct roots each with multiplicity a . Therefore $N(u, v) = A^a(u, v)$. Similarly, $\pi^{-1}(1)$ consists of the roots of $N(u, v) - D(u, v)$. By the assumptions on π this must have $k_2 = \frac{d}{b}$ distinct roots each of multiplicity b and so equals $B^b(u, v)$, implying $D(u, v) = A^a(u, v) + B^b(u, v)$. Finally, the roots of $D(u, v)$ are the points of $\pi^{-1}(\infty)$ and so by property G must have equal multiplicities; hence $D(u, v) = C^c(u, v)$ for some $c|d$.

Conversely, given the explicit form for π , successively set π equal to 0, 1, and ∞ , and solve. By assumption $A(u, v)$ and $B(u, v)$ have no repeated roots. Therefore $\pi^{-1}(0)$ is a set of $k_1 = \frac{d}{a}$ points of ramification index a , and $\pi^{-1}(1)$ is a set of $k_2 = \frac{d}{b}$ points of ramification index b . Likewise, the assumption that $D(u, v) = A^a(u, v) + B^b(u, v) = C^c(u, v)$ guarantees that the points of $\pi^{-1}(\infty)$ all have ramification index c (possibly equal to 1). Therefore by definition π has property G . \square

We want to enumerate all ν such that (ν, μ, π) satisfy all of our operating assumptions.

Summary of Assumptions:

1. $T = \{0, 1, \infty\}$ (arranged by automorphisms of \mathbb{P}^1)
2. ν , the monodromy data defining the DM local system $l_T \rightarrow \mathbb{P}^1 \setminus T$, satisfies $\sum \nu_j = 1$
3. π satisfies property G

4. μ is the monodromy data for the pull-back local system π^*l_T , such that:

- (a) $S = \pi^{-1}(\infty)$, that is, $\mu_i \notin \mathbb{Z}$ precisely for $s_i \in \pi^{-1}(\infty)$
- (b) the sum of the non-integral μ_i equals 2
- (c) Let m be the lowest common denominator of the μ_i . The ring of integers R in $\mathbb{Q}(\zeta_m)$ is either \mathcal{G} or \mathcal{E} .
- (d) μ satisfies INT or ΣINT

Corollary 11. *There are five triples (a, b, d) , corresponding to $\nu = (\frac{1}{a}, \frac{1}{b}, \frac{2}{d})$, which satisfy these assumptions:*

$$(3, 2, 12), (4, 2, 8), (6, 2, 6), (3, 3, 6), (4, 4, 4)$$

Proof. Because of property G , every point in the $\pi^{-1}(t_j)$ has the same ramification index. By the Proposition, they are integer multiples of a, b , and 1 respectively. The assumptions require $\mu_{0,i} = r_0 \cdot \nu_0$ and $\mu_{1,i} = r_1 \cdot \nu_1$ to all be integral, and furthermore $\sum_i \mu_{\infty,i} = \frac{d}{r_\infty} \nu_\infty = 2$. In order for the ring R to be \mathcal{G} or \mathcal{E} , the lowest common denominator of the ν_i must be one of 2, 3, 4, 6. Finally, using the fact $\sum_j \nu_j = 1$, one enumerates the solutions, which yields the above list of five. It is useful to note these are precisely the solutions of

$$\frac{1}{a} + \frac{1}{b} + \frac{2}{d} = 1.$$

Because the non-integral μ_i are attached to $S = \pi^{-1}(\infty)$, one sees the μ_i are all equal weight and hence certainly satisfy ΣINT , and indeed correspond to one of the ancestral examples or its equal weight descendants. \square

Now we want to classify, under above the assumptions, when the subspace \mathcal{S}_π is actually a subball quotient. By Theorem 6, given any valid ν , this amounts to a dimension count. The fact that $|T| = 3$ makes this easy to check.

Corollary 12. *The triples (a, b, d) from Corollary 11 all define \mathcal{S}_π that are codimension 1 subball quotients of $DM(d, \mu)$, where $\mu_i = \frac{2}{d}, \forall i$.*

Proof. Because $|T| = 3, \dim_{\mathbb{C}}(IH_1(\mathbb{P}^1, l_T)) = 1$. It is easy to check that the image of π^* is non-trivial: simply pull-back the generator $I_{1,\infty}$ and observe the resulting linear combination of basis elements in the lift is never the identity. Thus the image of π^* is one-dimensional.

By Theorem 6, the \mathcal{S}_π are subball quotients when the dimension count agrees. Here $\dim(\pi^*)^\perp$ is codimension 1. Note that $|S| = d$. Then by Proposition 18 the codimension of \mathcal{S}_π is determined by the number of ramification conditions imposed by the T -ramification class of π : precisely, $(d-3) - (2d-2-\frac{d}{a}(a-1)-\frac{d}{b}(b-1)) = (d-3)+2-d(\frac{1}{a}+\frac{1}{b}) = d(1-\frac{1}{a}-\frac{1}{b})+2-3$, which, using relation 4.4 above, is simply $d(\frac{2}{d}) - 1 = 1$, as desired. Note that one could also do a direct dimension count from the explicit form of π in this case. \square

Remark 18. The case $(3,2,12)$ corresponds to the moduli space of rational elliptic surfaces. See Corollary 14.

4.4.1 Some moduli spaces of inhomogeneous forms and ball quotients

Throughout let $A(u, v)$ and $B(u, v)$ be polynomials of degree d_1 and d_2 , respectively, with $d_1 < d_2$. The definitions can be extended to any number of polynomials, but we will use only two.

Definition 20. *Given the data (A, B) as above, let a, b and N be positive integers such that $d_1 a = N = d_2 b$. The choice of a and b determines a morphism $\Delta : \mathbb{WP}^n(d_1, d_2) \rightarrow \mathbb{P}^N$, given by $[A, B] \mapsto [A^a + B^b]$. We call such a map a pseudo-discriminant.*

Remark 19. Note the map is well-defined. Indeed, it is clear that the map $(A, B) \mapsto A^a + B^b$ is \mathbb{C}^* -equivariant, where \mathbb{C}^* acts as multiplication by $(\lambda^{d_1}, \lambda^{d_2})$ and λ^N respectively.

Remark 20. One can interpret this map in the language of the GKZ theory of resultants and discriminants for toric varieties [12]. There it appears as an “ A -discriminant”, with A an appropriately chosen set of homogeneous polynomials, before quotienting out by an associated group of toric automorphisms.

The question we ask is essentially the following elementary (but in many instances surprisingly rich) one.

Question: Given a degree N polynomial in two variables, when, and in how many ways, can it be written as the sum of an a^{th} power of a degree d_1 polynomial and a b^{th} power of a degree d_2 polynomial?

The “when” is the image of Δ and the “in how many ways” is the degree of Δ . To be more precise, we are interested in the number of solutions for a generic point in the image of Δ , not a complete analysis of the number of solutions for any given degree N polynomial.

Let ζ_m represent a primitive m^{th} root of unity. It is clear that $\Delta([\zeta_a^j A, \zeta_b^k B]) = [A^a + B^b] = \Delta([A, B])$. This is an obvious obstruction to the generic injectivity of Δ .

Proposition 20. *For a given a and b , Δ is generically at least $\gcd(a, b)$ -to-one. In particular, for Δ to be generically injective, it is necessary that $\gcd(a, b) = 1$.*

Proof. Because Δ is a map from weighted projective space $\mathbb{WP}^n(d_1, d_2)$ to projective space \mathbb{P}^N , one must check which pairs $(\zeta_a^j A, \zeta_b^k B)$ are equivalent under the weighted \mathbb{C}^* action.

Clearly it suffices to check when there exists a complex number λ such that simultaneously $\lambda^{d_1} = \zeta_a^j$ and $\lambda^{d_2} = \zeta_b^k$. In particular, λ must be an N^{th} root of unity. The two conditions are equivalent to asking for solutions to the following system of congruences:

$$\begin{aligned} d_1 x &\equiv d_1 j \pmod{N} &\longleftrightarrow & x \equiv j \pmod{a} \\ d_2 x &\equiv d_2 k \pmod{N} &\longleftrightarrow & x \equiv k \pmod{b} \end{aligned}$$

The Chinese Remainder Theorem implies there is a solution for all j and k precisely when a and b are relatively prime. More generally, it implies that for any given j there are $N/\gcd(a, b)$ values of k which lie in the same \mathbb{C}^* -orbit, so there are at least $N/(N/(\gcd(a, b))) = \gcd(a, b)$ distinct points mapped to the same point by Δ . \square

What follows is a sufficient condition for the degree of the pseudo-discriminant to be precisely $\gcd(a, b)$. Under this circumstance, the sole obstruction to injectivity is the one above, i.e., whether rescaling (A, B) by relevant roots of unity produces points in the same weighted \mathbb{C}^* -orbit.

Proposition 21. *Assume $d_2 > d_1 + 1$ and $b = 2$. Then $\Delta(A_1, B_1) = \Delta(A_2, B_2)$ implies $A_1^a = A_2^a$ (equivalently, $B_1^b = B_2^b$), that is, $A_1 = \zeta_a^j A_2$ and $B_1 = \zeta_b^k B_2$. Furthermore, the degree of Δ is $\gcd(a, b)$.*

Proof. The argument we give is inspired by [28, p. 17]. Consider the space of polynomial quadruples (A_1, B_1, A_2, B_2) subject to the constraint that $A_1^a + B_1^b = A_2^a + B_2^b$. Remove the subset $A_1^a = A_2^a$ (equivalently, $B_1^b = B_2^b$). What remains are the solutions to $\Delta(A_1, B_1) = \Delta(A_2, B_2)$ other than $A_1 = \zeta_a^j A_2$ and $B_1 = \zeta_b^k B_2$. Call this set Q_Δ . We claim Q_Δ is empty.

We argue by contradiction. Assume it is not empty. Then it has a dimension. The dimension cannot be any less than the dimension of the space of polynomials (A_1, B_1) , which is $(d_1 + 1) + (d_2 + 1) = d_1 + d_2 + 2$. So $\dim_{\mathbb{C}}(Q_\Delta) \geq d_1 + d_2 + 2$.

But there is another way to count the dimension of Q_Δ . Rewrite the defining constraint as $A_1^a - A_2^a = B_2^b - B_1^b$. Because $b = 2$, the right hand side of the equation factors as $(B_2 - B_1)(B_2 + B_1)$. Specifying the pair (A_1, A_2) determines $(B_2 - B_1)(B_2 + B_1)$. Because $A_1^a \neq A_2^a$, $A_1^a - A_2^a$ has N roots (counting multiplicity). By assigning these roots to each of $(B_2 - B_1)$ and $(B_2 + B_1)$, these factors are completely determined up to relative scaling and the finite ambiguity in assigning the roots. Thus the dimension of the set of solutions Q_Δ is the dimension of (A_1, A_2) plus one to account for the relative scaling. That is, $\dim_{\mathbb{C}}(Q_\Delta) = 2(d_1 + 1) + 1 = 2d_1 + 3$.

One concludes $2d_1 + 3 \geq d_1 + d_2 + 2$, hence $d_1 + 1 \geq d_2$, or equivalently $d_1 + 1 > d_2$. But this contradicts the assumption of our theorem that $d_2 > d_1 + 1$. Thus Q_Δ must be empty. \square

Remark 21. Although the $b = 2$ condition can be relaxed, the $d_2 > d_1 + 1$ is necessary. As an example, when $d_1 = 2, d_2 = 3, N = 6$, Δ is generically a 40-to-1 map [11].

Proposition 22. *The pseudo-discriminant Δ is $SL_2(\mathbb{C})$ -equivariant. It descends to a map of GIT quotients.*

Proof. The $SL_2(\mathbb{C})$ -equivariance is immediate, because it acts through the standard representation on $V \cong \mathbb{C}^2$ in each case: the domain is $\mathbb{WP}^n(\text{Sym}^{d_1}(V) \oplus \text{Sym}^{d_2}(V))$ and the range is $\mathbb{P}(\text{Sym}^N(V))$. \square

One should think of the image of this map as lying inside the moduli space of N unordered points. The domain and range both offer potentially different compactifications for the open set. In particular, for $N = 12$ or $N = 8$ the compactification of the image is a Baily-Borel compactification for the Eisenstein or Gaussian ancestral examples respectively. Thus there is an alternate compactification to the GIT compactification for certain weighted projective space quotients.

Observe this gives alternate description of the \mathcal{S}_π .

Theorem 7. *The classification of \mathcal{S}_π satisfying property G in Corollary 12 is identical to the classification of pseudo-discriminants with image a hypersurface of codimension 1.*

Proof. This is simply a dimension count. The condition that the image of Δ be a hypersurface is the statement that $(d_1 + 1) + (d_2 + 1) = N$, where $N = ad_1 = bd_2$. Divide by N to get $\frac{1}{a} + \frac{1}{b} + \frac{2}{N} = 1$, which is the same constraint as the one we discovered in the classification of π with property G. \square

Corollary 13. *The moduli spaces of inhomogeneous binary forms of bidegree (a, b) , for (a, b) taken from the list in the above theorem, are branched covers of subball quotients of the corresponding $DM(n, \mu)$. For one of the cases, $(3, 2, 12)$, Δ is an embedding on a suitable open subset, and for two others, $(6, 2, 6)$ and $(4, 2, 8)$, it is generically 2-to-1.*

Remark 22. This result parallels the statement that the ancestral examples, thought of as moduli spaces of binary forms of degree 8 and 12, are ball quotients.

Corollary 14. *The moduli space of rational elliptic surfaces is a ball quotient, in particular it is a hyperball quotient of the Eisenstein ancestral example.*

Proof. The example $(3, 2, 12)$ is the GIT moduli space of rational elliptic surfaces presented as rational Weierstrass fibrations. This GIT description of the moduli space was first discovered by Miranda [22], following Mumford. \square

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